Derivatives and Differentials

1 Derivatives

1.1 Ordinary Derivatives

In Calculus 1 we introduced the derivative. If $y = f(x)$ then we defined the derivative as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1}$$

The process can be long but we also introduced rules which enabled us to calculate these derivative much faster. So, for example, if

$$y = \frac{x}{x^2+1} \tag{2}$$

then by the quotient rule we have

$$y' = \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \tag{3}$$

The process of going from the derivative (3) back to the original function (2) we use integration.

Example 2 Consider the following implicit function

$$x^2 + 3xy - y^4 = 3x + 2y - 2. \tag{4}$$

By implicit differentiation we have

$$2x + 3(1 \cdot y + x \cdot y') - 4y^3 y' = 3 + 2y'$$

and solving for y' gives

$$y' = -\frac{2x+3y-3}{3x-4y^3-2}. \tag{5}$$

Note: Equation (5) is a first order ODE. Equation (4) is a solution of this ODE. Could we solve (5) giving (4)?

1.2 Partial Derivatives

In Calculus 3 we consider functions of more than one variable and, in particular, functions like $z = f(x, y)$. We ask, can we take derivatives of these. We introduced two derivatives, an x derivative and a y derivative defined by

$$\begin{aligned} \frac{\partial z}{\partial x} &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \\ \frac{\partial z}{\partial y} &= \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}. \end{aligned} \tag{6}$$

In each, we only vary one variable holding the other fixed (treated as a constant) so we can use the rules from calculus 1. The following examples illustrate.

Example 3 Consider

$$z = x^3y^2 \tag{7}$$

For the x derivative, we hold y fixed (we will replace y with a c for now), so

$$z = x^3c^2 \tag{8}$$

so

$$z_x = 3x^2c^2 = 3x^2y^2 \tag{9}$$

Similar for the y derivative, we hold x fixed (we will replace x with a c for now), so

$$z = c^3y^2 \tag{10}$$

so

$$z_y = c^3 3y = 3x^3y. \tag{11}$$

The subscripts of x and y are notation for the x and y derivatives.

Example 4 Consider

$$x^3 + y^3 + z^3 + 6xyz = 1. \tag{12}$$

Here, z is defined implicitly and we need to use implicit differentiation. so

$$3x^2 + 3z^2z_x + 6yz + 6xyz_x = 0 \tag{13}$$

and

$$3y^2 + 3z^2z_y + 6xz + 6xyz_y = 0 \tag{14}$$

and solving for z_x and z_y gives

$$z_x = -\frac{x^2 + 2yz}{z^2 + 2xy}, \quad z_y = -\frac{y^2 + 2xz}{z^2 + 2xy}. \tag{15}$$

2 Differentials

If we consider approximating the change in y by moving a small amount in x, we can use the equation of the tangent. At the point (a, b), the equation of the tangent is

$$y - b = f'(a)(x - a). \tag{16}$$

Now if we let $x = a + dx$ and $y = b + dy$, we see from (16) that

$$b + dy - b = f'(a)(a + dx - a),$$

or

$$dy = f'(a)dx,$$

a relation between the differential dx and dy. We go further and define this relationship for general x as

$$dy = f'(x)dx$$

which applies for all x. So, for example, if $y = x^2$ then

$$dy = 2x\,dx$$

If $y = xe^x$ then

$$dy = (xe^x + e^x)\,dx$$

Does this extend to $3 - D$? Yes. We now follow the tangent plane. The tangent plane is given by

$$z - c = f_x(a, b)(x - a) + f_y(a, b)(y - b). \tag{17}$$

Now if we let $x = a + dx$, $y = b + dy$ and $z = c + dz$ then from (17) we see that

$$c + dz - c = f_x(a, b)(a + dx - a) + f_y(a, b)(b + dy - b).$$

or

$$dz = f_x(a, b)\,dx + f_y(a, b)\,dy,$$

a relation between the differential dx, dy and dz. We go further and define this relationship for general x and y as

$$dz = f_x\,dx + f_y\,dy,$$

or

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

Example 7
If $z = x^2y^5$, find dz. Calculating the partial derivatives, we find that

$$\frac{\partial z}{\partial x} = 2xy^5 \quad \frac{\partial z}{\partial y} = 5x^2y^4$$

so the differential dz is

$$dz = 2xy^5\,dx + 5x^2y^4\,dy.$$

Example 8

If $z = e^{xy} + x\sin y$, find dz. Calculating the partial derivatives, we find that

$$\frac{\partial z}{\partial x} = ye^{xy} + \sin y \quad \frac{\partial z}{\partial y} = xe^{xy} + x\cos y$$

so the differential dz is

$$dz = \left(ye^{xy} + \sin y\right)\, dx + \left(xe^{xy} + x\cos y\right)\, dy.$$

3 Exact ODEs

Consider the following. Suppose we have

$$x^2 - y - xy^3 = c \tag{18}$$

where c is some constant. If we set

$$z = x^2 - y - xy^3 \tag{19}$$

then the differential dz is

$$dz = (2x - y^3)dx + (-1 - 3xy^2)dy. \tag{20}$$

Suppose we where told that

$$(2x - y^3)dx + (-1 - 3xy^2)dy = 0. \tag{21}$$

Then from (20) $dz = 0$ giving that $z = c$ and since $z = x^2 - y - xy^3$ from (19) then we recover (18)

This is the basic idea on how to solve *exact* ODEs.