Infinite Series

Special Series

#1 Geometric

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

converges if $|r| < 1$ and if so, $S_\infty = \dfrac{a}{1-r}$.

#2 Telescopic

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - b_{n+1}$$

(i) stop the series at N,
(ii) write out the terms and cancel,
(iii) $N \to \infty$

#3 Harmonic

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

The series diverges.

#4 p- series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \begin{array}{ll} p < 1 & \text{diverges} \\ p > 1 & \text{converges} \end{array}$$

Tests

#1 n^{th} Term Test

$$\text{If} \quad \lim_{n\to\infty} a_n \neq 0 \tag{1}$$

the series diverges.

#2 Integral Test $a_n = f(n)$
First check the conditions:

(i) $f(x) \geq 0$,
(ii) $f(x)$ is continuous, and
(iii) $f(x)$ is decreasing ($f'(x) < 0$)

If

$$\int_1^{\infty} f(x)dx \text{ converges (diverges)} \quad \text{then} \quad \sum_{n=1}^{\infty} a_n \text{ converges (diverges).} \tag{2}$$

#3 Limit Term Test (LCT) If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \# \text{ (not zero)} \tag{3}$$

both series do the same, either converge or diverge.

#4 Direct Comparison Term Test (DCT)

Suppose $0 \le a_n \le b_n$. If

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \text{ diverges} \quad \text{then} \quad \sum_{n=1}^{\infty} b_n \text{ diverges} \\ \sum_{n=1}^{\infty} b_n \text{ converges} \quad \text{then} \quad \sum_{n=1}^{\infty} a_n \text{ converges} \end{aligned} \tag{4}$$

#5 Ratio Test

$$\text{If} \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \text{L.} \quad \text{If} \quad \begin{array}{l} L > 1 \text{ diverges} \\ L < 1 \text{ converges} \\ L = 1 \text{ no conclusion} \end{array} \tag{5}$$

#6 Root Test

$$\text{If} \quad \lim_{n\to\infty} \sqrt[n]{a_n} = \text{L.} \quad \text{If} \quad \begin{array}{l} L > 1 \text{ diverges} \\ L < 1 \text{ converges} \\ L = 1 \text{ no conclusion} \end{array} \tag{6}$$

#7 Alternating Series Term Test (AST)

For series of the form

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n \quad a_n \ge 0$$

If

$$\begin{aligned} (i) \quad & a_{n+1} < a_n \text{(decreasing)} \\ (ii) \quad & \lim_{n\to\infty} a_n = 0 \end{aligned} \tag{7}$$

the series converges.

Absolute and Conditional Convergence ($a_n \ge 0$)

If $\sum_{n=1}^{\infty} (-1)^n a_n$ converges and $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges conditionally.

If $\sum_{n=1}^{\infty} (-1)^n a_n$ converges and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely.