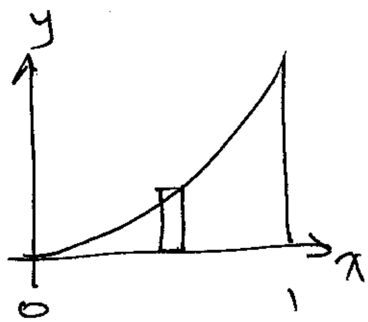
Areas

So we introduced area under curves and approximated these by rectangles. Today we consider some more examples

Ex1 Find the area under

$f(x) = x^2$ on $[0, 1]$



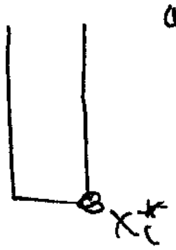
what we will do is construct a typical rectangle (i^{th} rectangle) add them up & let # rect $\to \infty$

(1) subdivide interval

so the thickness of each rect.

$$\Delta x = \frac{1}{n}$$

(2) For the i^{th} rectangle, pick the right point and use this for the height so $x_i^* = \frac{i}{n}$



$$h_i = f(x_i^*) = \left(\frac{i}{n}\right)^2$$

(3) The area A_i of this rectangle is

$$A_i = f(x_i^*)\Delta x$$

$$= \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{i^2}{n^3}$$

(4) Add up rectangles

$$A \simeq \sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$$

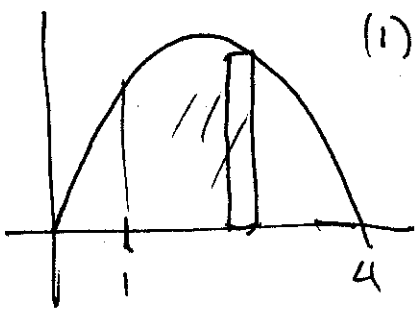
$$= \frac{2}{6} + \frac{3}{6n} + \frac{1}{6n^2}$$

Now $\lim_{n\to\infty} A = \lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$

$$= \frac{1}{3}$$

Ex 2 Find the area under $f(x) = 4x - x^2$

on $[1, 4]$



(1) subdivide into n pieces $\Delta x = \frac{4-1}{n} = \frac{3}{n}$

(2) right endpt of i^{th} rectangle

$$x_i^* = 1 + \frac{3i}{n}$$

(3) height of i^{th} rectangle

$$h_i = f(x_i^*) = 4\left(1+\frac{3i}{n}\right) - \left(1+\frac{3i}{n}\right)^2$$

$$=$$

(4) Area of i^{th} rectangle

$$A_i = \left[4\left(1+\frac{3i}{n}\right) - \left(1+\frac{3i}{n}\right)^2\right]\frac{3}{n}$$

(5) Add up rectangles

$$A \simeq \sum_{i=1}^{n} \left[4\left(1+\frac{3i}{n}\right) - \left(1+\frac{3i}{n}\right)^2\right]\frac{3}{n}$$

(6) $N \to \infty$

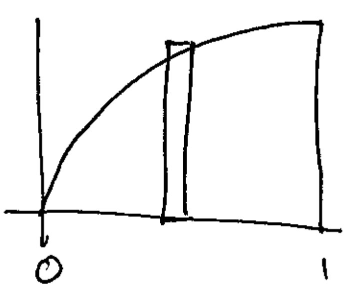
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[4\left(1+\frac{3i}{n}\right) - \left(1+\frac{3i}{n}\right)^2\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3+\frac{6i}{n} - \frac{9i^2}{n^2}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3\cdot 3n}{n} + \frac{6n(n+1)}{2n}\cdot\frac{3}{n} - \frac{27\, n(n+1)(2n+1)}{6n^3}$$

$$= 9 + \cancel{9} - \cancel{9} = 9$$

Ex3 $f(x) = \sqrt{x}$ on $[0,1]$



(1) subdivide interval into n

$\Delta x = \frac{1}{n}$

(2) $x_i^* = \frac{i}{n}$

(3) $h_i = f(x_i^*) = \sqrt{\frac{i}{n}}$

(4) $A_i = \sqrt{\frac{i}{n}}\,\frac{1}{n}$

(5) Add $A \approx \sum_{i=1}^{n} \sqrt{\frac{i}{n}}\,\frac{1}{n} = \sum_{i=1}^{n} \frac{\sqrt{i}}{n\sqrt{n}}$

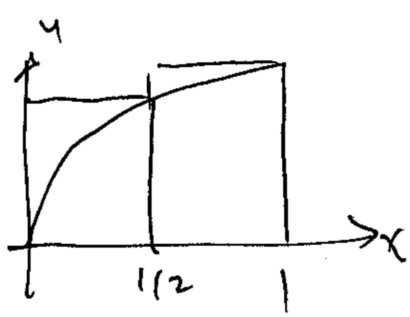
(6) $n \to \infty$ $A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sqrt{i}}{n\sqrt{n}}$

Unfortunately, we don't have a formula for

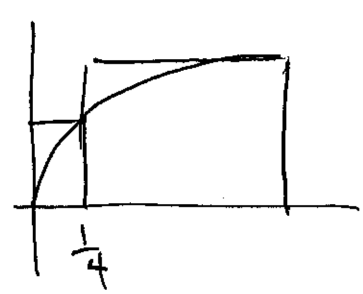
$$\sum_{i=1}^{n} \sqrt{i}$$

So we want to rethink this

Instead of having equally space rectangles let the thickness change. In particular let the right endpt of each rectangle be different



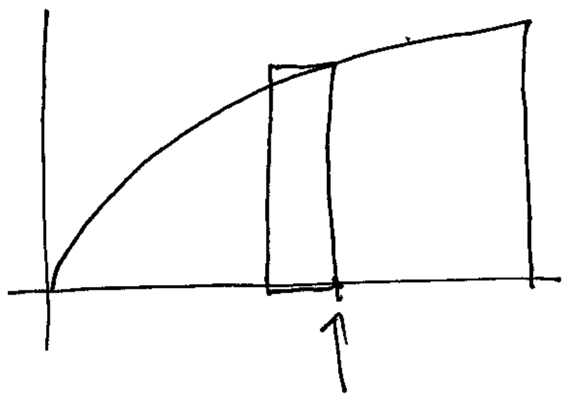
before



now

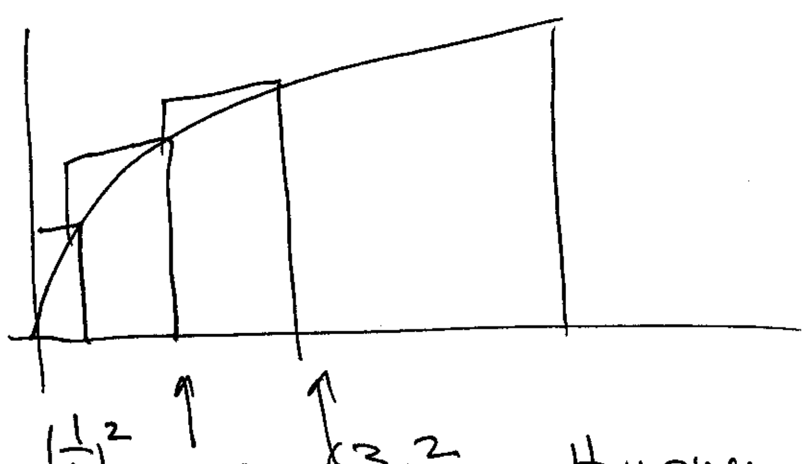
why is this better? B/c for the height

$\sqrt{\frac{1}{4}} = \frac{1}{2}$



choose $x_i^* = \left(\frac{i}{n}\right)^2$

the first few



$\left(\frac{1}{n}\right)^2$ $\left(\frac{2}{n}\right)^2$ $\left(\frac{3}{n}\right)^2$

However, each thickness is different

$h_1 = \sqrt{\left(\frac{1}{n}\right)^2} = \frac{1}{n}$

$h_2 = \sqrt{\left(\frac{2}{n}\right)^2} = \frac{2}{n}$

$h_3 = \sqrt{\left(\frac{3}{n}\right)^2} = \frac{3}{n}$

$\Delta x_1 = \frac{4}{n^2} - \frac{1}{n^2} = \frac{3}{n^2}$

$\Delta x_2 = \frac{9}{n^2} - \frac{4}{n^2} = \frac{5}{n^2}$

So in general

$$\Delta x_i = x_i^* - x_{i-1}^*$$

$$= \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} = \frac{i^2 - (i^2 - 2i + 1)}{n^2}$$

$$= \frac{2i-1}{n^2}$$

Now $A_i = f(x_i^*)\,\Delta x_i$

$$= \frac{i}{n} \cdot \frac{2i-1}{n^2}$$

$$A = \lim_{n \to \infty} \sum_{n=1}^{n} \frac{2i^2 - i}{n^3}$$

$$= \lim_{n \to \infty} \frac{2}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{1}{n^3} \frac{n(n+1)}{2}$$

$$= \frac{4}{6} = \frac{2}{3}$$

So changing the thickness of each rectangle helps us here