What is the Right Solution Concept for No-Limit Poker?

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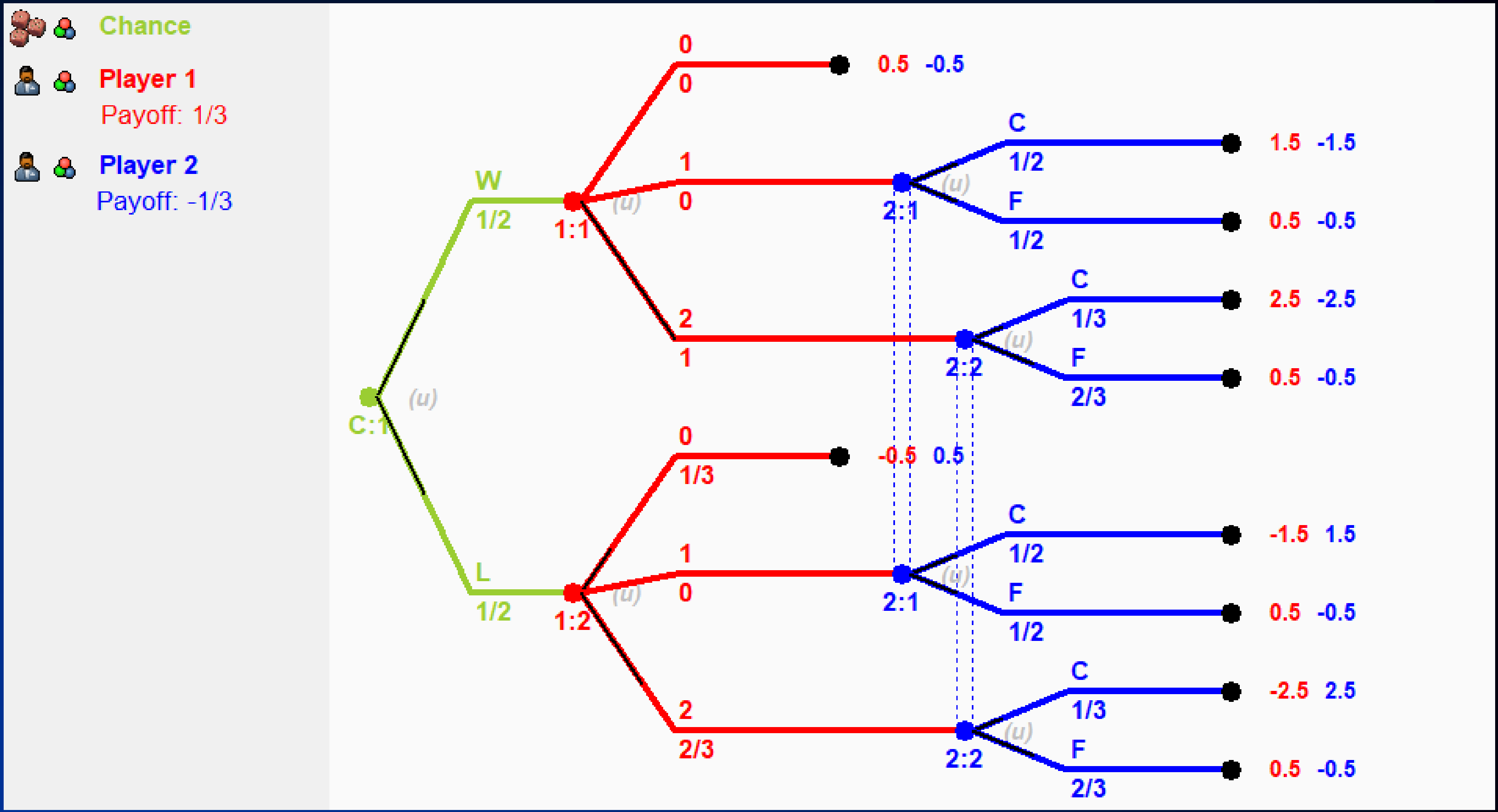
No-limit clairvoyance game

- Player 2 is given no private cards
- Player 1 is given single card drawn from a distribution that is half winning hands and half losing hands.
- Both players have stack size of n and both ante \$0.50.
- P1 is allowed to bet any amount x in [0,n].
- Then P2 is allowed to call or fold (but not raise).

- (Note that this is equivalent to if P1 is dealt 50-50 K or J and P2 is dealt Q with probability 1).
- P1 is "clairvoyant" in that he "knows" what P2 has.

No-limit clairvoyance game



Solution to clairvoyance game

- Game can be solved analytically.

- P1 bets n with prob. 1 with winning hand.
- P1 bets n with prob. n/(1+n) with losing hand
- P1 bets 0 otherwise with losing hand
- For all x in [0,n], P2 calls a bet of size x with probability 1/(1+x).

Solution to clairvoyance game

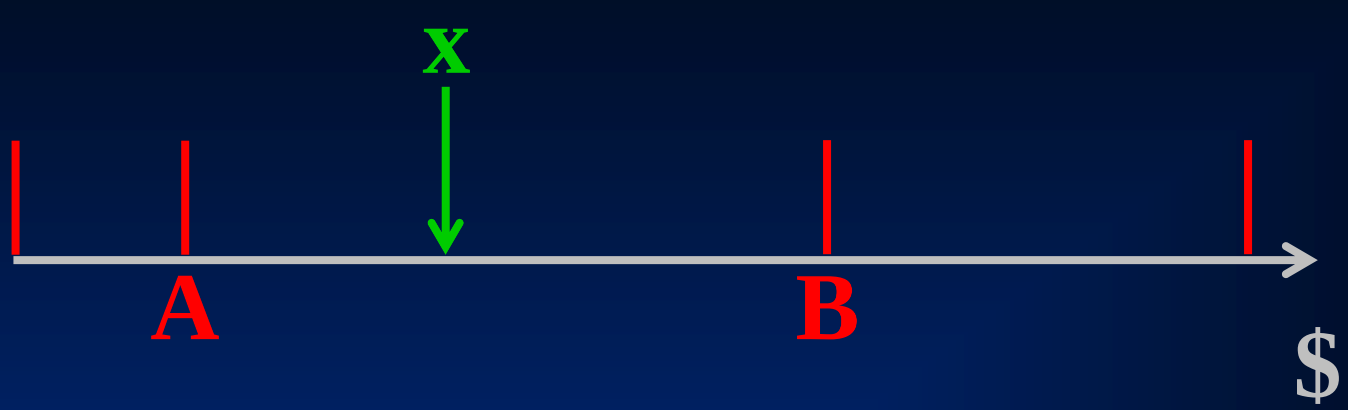
Proposition 1. *The strategy profile presented above is a Nash equilibrium of the clairvoyance game.*

Proof. Player 2 must call a bet of size x with probability $\frac{1}{1+x}$ in order to make player 1 indifferent between betting x and checking with a losing hand. For a given x, player 1 must bluff $\frac{x}{1+x}$ as often as he value bets for player 2 to be indifferent between calling and folding. Given these values, the expected payoff to player 1 of betting size x is $v(x) = \frac{x}{2(1+x)}$. This function is monotonically increasing, and therefore player 1 will maximize his payoff with $x = n$. $\square$

Application to no-limit Texas hold ‘em

- Despite the game’s simplicity, the solution has been applied to interpret bet sizes for the opponent that fall outside an abstracted game model by many of the strongest agents for full no-limit Texas hold ‘em
  - Ganzfried & Sandholm IJCAI ‘13, Jackson ’13
- E.g., if the “betting abstraction” allows for fold, call, bet pot, allin, and opponent bets 3*pot or 0.5*pot.

Action translation



- $f_{A,B}(x) \equiv$ probability we map x to A
  - Will also denote as just $f(x)$

[Ganzfried & Sandholm IJCAI-13]

Pseudo-harmonic mapping

- Maps opponent's bet x to one of the nearest sizes in the abstraction A, B according to:
- $f(x) = \frac{(B-x)(1+A)}{(B-A)(1+x)}$
- f(x) is probability that x is mapped to A
- Example: suppose opponent bets 100 into pot of 500, and closest sizes are "check" (i.e., bet 0) or to bet 0.25 pot. So A = 0, x = 0.2, B = 0.25.
- Plugging these in gives f(x) = 1/6 = 0.167.

All Nash equilibria in clairvoyance game

It turns out that player 2 does not need to call a bet of size $x \neq n$ with exact probability $\frac{1}{1+x}$: he need only not call with such an extreme probability that player 1 has an incentive to change his bet size from n to x (with either a winning or losing hand). In particular, it can be shown that player 2 need only call a bet of size x with any probability (which can be different for different values of x) in the interval $\left[\frac{1}{1+x}, \min\left\{\frac{n}{x(1+n)}, 1\right\}\right]$ in order to remain in equilibrium. Only the initial equilibrium is reasonable, however, in the sense that we would expect a rational player 2 to maintain the calling frequency $\frac{1}{1+x}$ for all x so that he plays a properly-balanced strategy in case player 1 happens to bet x.

Responding to opponent's mistake

- Suppose the opponent bets $x < n$, as opposed to the "optimal" size n that he should bet in equilibrium.
- How should we interpret this, what should we deduce, how should we reply (i.e., how often should we call?)

Responding to opponent’s mistake

- If he is betting x instead of n, it seems like he isn’t aware of the equilibrium. If he was, then he would have bet the unique optimal size n.
- He probably thinks mistakenly that x is optimal size.
- Let’s give him benefit of the doubt that, given this mistake in selecting bet size, he is still playing a rational strategy that uses the suboptimal size x.
- In particular, suppose he is playing a Nash equilibrium in the game where he is restricted to only betting x (or to betting 0, i.e., *checking*).

Natural solution

- This analysis leads to the equilibrium that I have singled out – the one where player 2 calls with probability $1/(1+x)$ vs. all bet sizes x.
- The other equilibria all play heed to the concern that the opponent may try to exploit us by deviating to bet x instead of n.
- But we do not need to be as concerned about this, since a rational opponent who knew to bet n would not be betting x!!

Traditional equilibrium refinements

- This equilibrium I have singled out is not the one specifically selected for by any popular refinement.
- Infinite-many equilibria each satisfy the following:
  - Extensive-form trembling-hand perfect equilibrium
  - Normal-form trembling-hand perfect equilibrium
  - Extensive-form proper equilibrium
- (Note that the normal-form vs. extensive-form versions are incomparable for a given solution concept).

Normal-form proper equilibrium

- There is a unique normal-form proper equilibrium, but it does not correspond to the "natural" solution I have singled out.
- For the 0/1/2 game the unique NFPE calls vs. a bet size of 1 with probability 5/9, while the "natural" solution calls with probability ½.
  - Thank you to Troels Sorensen and Jiri Cermak for assisting with this computation.

Concluding questions

- What is the “right” solution in this game?
- Does the “intuitive solution” have any justification?
- Are existing equilibrium refinement concepts useful for this game?
- Can there be a new refinement concept that agrees with the “intuitive solution”?