Math 3331 – ODE's

Now we consider 2nd order linear ODE's.
Most of the ideas extend to higher order

2nd Order Linear

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

y, y', y'' power 1

ex $y'' + 2y' + y = 0$ ✓

$x^2 y'' + y = 10x$ ✓

$y'' + y^2 = 0$ no

$y'' + \underbrace{yy'}_{} = 0$ no

we pretty much cannot solve 2nd order
ODE's in general.

we assume that $a_2(x) \neq 0$

so $$y'' + \frac{a_1(x)}{a_2(x)} y' + \frac{a_0(x)}{a_2(x)} y = \frac{b(x)}{a_2(x)}$$

let $p(x) = \frac{a_1(x)}{a_2(x)}$ $\quad q(x) = \frac{a_0}{a_2}$ $\quad g(x) = \frac{b(x)}{a_2(x)}$

so $$y'' + p(x)y' + q(x)y = g(x)$$ standard form linear

if $g(x) = 0$ ODE is homogeneous

$g(x) \neq 0$ " non homogeneous

the sol^n of

$$y'' + p(x)y' + q(x)y = 0$$

is $y = c_1 y_1 + c_2 y_2$

where y_1 & y_2 are 2 linearly independent

sol^n

If y_p is a particular sol^n of

$$y'' + p(x)y' + q(x)y = g(x)$$

General sol^n of non homogeneous ODE

is $\quad y = y_c + y_p$

$y_c = c_1 y_1 + c_2 y_2$ complementary sol^n

The goal of the next couple of weeks is to find the general sol^n of 2nd order linear ODEs

Ex $\quad y'' - 3y' + 2y = 0$

Two sol^n's are $y_1 = e^x$, $y_2 = e^{2x}$

check $\quad y_1' = e^x$, $y_1'' = e^x \qquad y_2' = 2e^{2x}$, $y_2'' = 4e^{2x}$

LS. $\quad e^x - 3e^x + 2e^x = 0$ ✓

LS $\quad 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$ ✓

so sol^n of $y'' - 3y' + 2y = 0$

is $y = c_1 e^{x} + c_2 e^{2x}$

<u>Linear independe</u>

if $c_1 y_1 + c_2 y_2 = 0$ only if $c_1 = c_2 = 0$

then y_1 & y_2 are linearly indep

previous ex.

$$c_1 e^{x} + c_2 e^{2x} = 0 \quad \text{for all } x$$

then certainly for $x = 0$

$$c_1 e^{0} + c_2 e^{0} = 0 \Rightarrow c_1 + c_2 = 0$$

$$c_2 = -c_1$$

$$c_1 e^{x} - c_1 e^{2x} = 0$$

$$c_1 (e^{x} - e^{2x}) = 0$$

sub $x=1$ $\quad c_1 (e^{1} - e^{2}) = 0 \Rightarrow c_1 = 0 \Rightarrow c_2 = 0$ ✓

Define Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

if $W(y_1, y_2) = 0$ then y_1 y_2 are linearly dep

if $W(y_1, y_2) \neq 0$ in general (for some interval x)

then y_1, y_2 linearly indep

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x(2e^{2x}) - e^x e^{2x}$$

$$= e^{3x} \neq 0$$

so yes linear indep

Ex $\quad y'' + y = x^2 + 2$

Show $y_p = x^2$ is a particular sol^n

$y_p' = 2x, \quad y_p'' = 2$

L.S. $y'' + y = 2 + x^2 = $ R.S. so yes

The sol^n of

$$y'' + y = 0$$

is $\quad y_c = c_1 \sin x + c_2 \cos x$

so the general sol^n is

$$y = y_c + y_p$$

$$= c_1 \sin x + c_2 \cos x + x^2$$

<u>Next Up</u>

Solving $\quad ay'' + by' + cy = 0$

a, b, c #'s