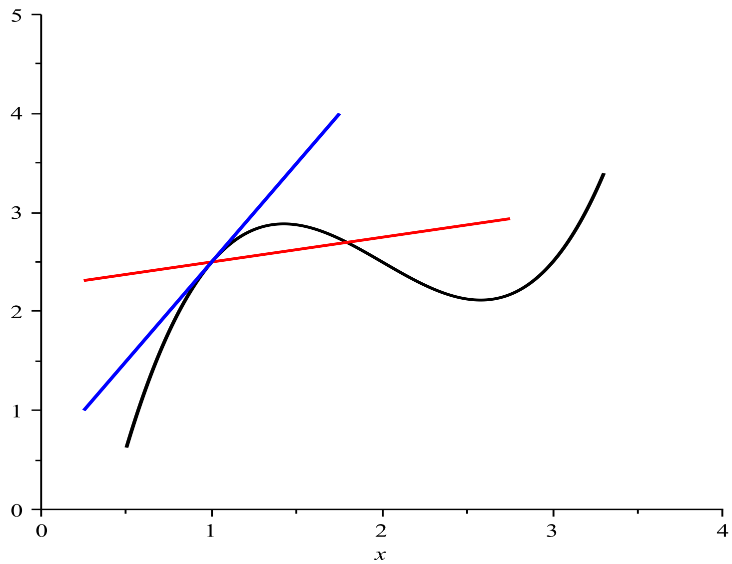
Calculus 3 - Partial Derivatives

In calculus 1 we introduced the derivative. We considered the function $y = f(x)$ and a secant to the curve that goes through the points $(x, f(x))$ and $(x+h, f(x+h))$ (in red). Then we let $h \to 0$ and the secant line (red) becomes the tangent line (blue)



Mathematically, we define the derivative of a function $y = f(x)$ as

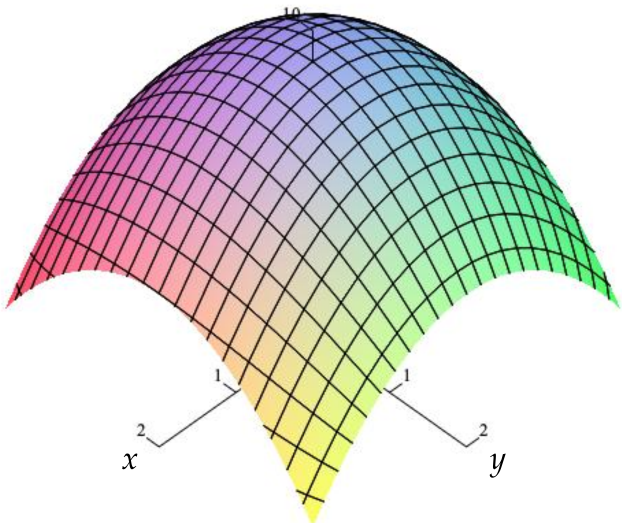
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

So can we define the derivative for functions of more than one independent variable? Here we will consider functions of two independent variables

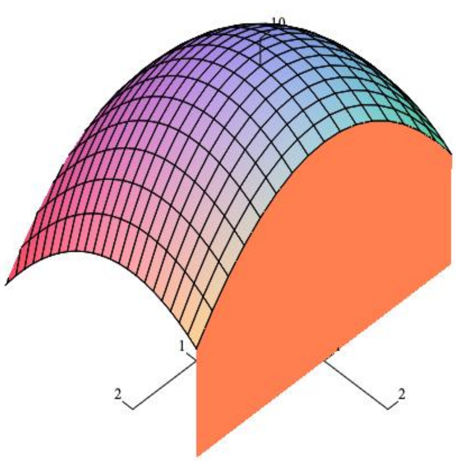
$$z = f(x, y) \tag{2}$$

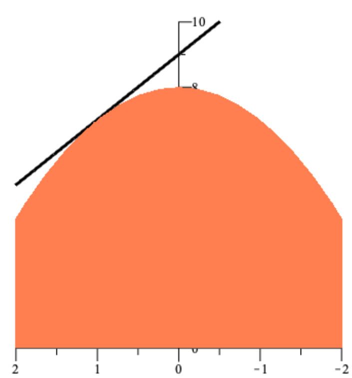
but these ideas certainly extended to an arbitrary number of independent variables.

Consider some surface $z = f(x, y)$



Here, we will take a slice where we fix y to some value and vary x (left pic). If we look straight down the y axis we see (right pic)

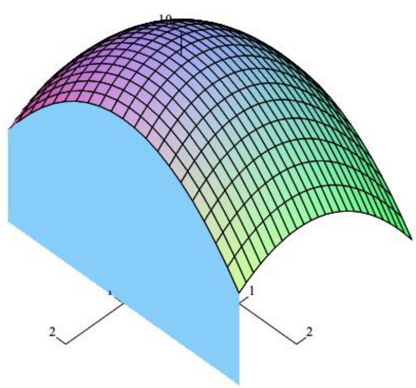


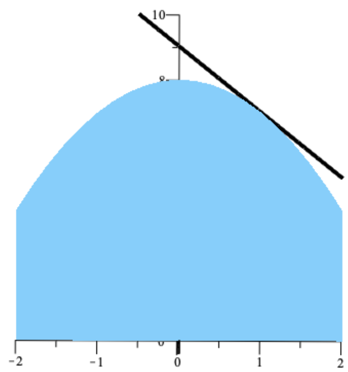


This certainly looks like something from Calc 1 and so we can find a tangent line. So mathematically we have

$$\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \tag{3}$$

Similarly, we fix x and vary y and look straight down the x axis we see Again, this looks like something from Calc 1 and so we can find a tangent





line. So mathematically we have

$$\lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k} \tag{4}$$

So now we define two different derivatives, an x derivative and the y derivative (called *partial derivatives*) and are defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}, \tag{5a}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}. \tag{5b}$$

Abbreviations

As we have abbreviations for ordinary derivatives like y' or f' we also have abbreviations for partial derivatives. These would be

$$\frac{\partial f}{\partial x} = f_x = z_x, \quad \frac{\partial f}{\partial y} = f_y = z_y \tag{6}$$

So let's look at an example. Consider

$$f(x, y) = 2x - y \tag{7}$$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \\ &= \lim_{h \to 0} \frac{(2(x+h) - y) - (2x - y)}{h} \\ &= \lim_{h \to 0} \frac{2x + 2h - y - 2x + y}{h} \\ &= \lim_{h \to 0} \frac{2h}{h} \\ &= 2\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}, \\ &= \lim_{k \to 0} \frac{2x - (y+k) - (2x - y)}{h} \\ &= \lim_{k \to 0} \frac{2x - y - k - 2x + y}{h} \\ &= \lim_{k \to 0} \frac{-k}{k} \\ &= -1\end{aligned} \tag{9}$$

so

$$f_x = 2, \quad f_y = -1. \tag{10}$$

One thing to note is that the partial derivatives are usually different!

Here's another example

$$f(x, y) = x^2 e^y \tag{11}$$

so

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \\
&= \lim_{h \to 0} \frac{(x+h)^2 e^y - x^2 e^y}{h} \\
&= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - x^2) e^y}{h} \\
&= \lim_{h \to 0} \frac{(2x+h) h e^y}{h} \\
&= \lim_{h \to 0} (2x+h) e^y \\
&= 2x e^y
\end{aligned} \tag{12}$$

So you're probably thinking - is there a short cut? Well yes. Since we defined these derivatives as fixing one variable and letting the other vary we are essential treating the fixed variable as constant so in the previous example

$$\frac{\partial f}{\partial x} = \frac{\partial \left(x^2 e^y \right)}{\partial x} = \frac{\partial \left(x^2 e^c \right)}{\partial x} = \frac{\partial \left(x^2 \right)}{\partial x} e^c = 2x e^c = 2x e^y \tag{13}$$

To calculate the y derivative we would

$$\frac{\partial f}{\partial y} = \frac{\partial \left(x^2 e^y \right)}{\partial y} = \frac{\partial \left(c^2 e^y \right)}{\partial y} = c^2 \frac{\partial \left(e^y \right)}{\partial y} = c^2 e^y = x^2 e^y \tag{14}$$

With some practice, you won't have to always replace the fixed variable with a c.

Example Calculate the partial derivatives for $z = \ln(x^2 + xy^4 + 1)$

Here we use the change rule first so

$$z_x = \frac{1}{x^2 + xy^4 + 1} \cdot (2x + y^4) \tag{15}$$

Similarly

$$z_y = \frac{1}{x^2 + xy^4 + 1} \cdot 4xy^3 \tag{16}$$

The Differential

If we consider approximating the change in y by moving a small amount in x, we can use the equation of the tangent. At the point (x_0, y_0), the equation of the tangent is

$$y - y_0 = f'(x_0)(x - x_0). \tag{17}$$

Now if we let $x = x_0 + dx$ and $y = y_0 + dy$, from (17) we see that

$$y_0 + dy - y_0 = f'(x_0)(x_0 + dx - x_0),$$

or

$$dy = f'(x_0)dx,$$

a relation between the differential dx and dy. We go further and define this relationship for general x as

$$dy = f'(x)dx,$$

or

$$dy = \frac{dy}{dx}\,dx,$$

which applies for all x. Does this extend to $3-D$? Yes. We now follow the tangent plane. The tangent plane is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{18}$$

Now if we let $x = x_0 + dx$, $y = y_0 + dy$ and $z = z_0 + dz$ then from (18) we see that

$$z_0 + dz - z_0 = f_x(x_0, y_0)(x_0 + dx - x_0) + f_y(x_0, y_0)(y_0 + dy - y_0).$$

or

$$dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy,$$

a relation between the differential dx, dy and dz. We go further and define this relationship for general x and y as

$$dz = f_x\,dx + f_y\,dy,$$

or

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

Example 1

If $z = x^2y^5$, find dz. Calculating the partial derivatives, we find that $\frac{\partial z}{\partial x} =$

$2xy^5$ and $\frac{\partial z}{\partial y} = 5x^2y^4$ so the differential dz is

$$dz = 2xy^5\,dx + 5x^2y^4\,dy.$$

Example 2

If $z = e^{xy} + x\sin y$, find dz. Calculating the partial derivatives, we find that $\frac{\partial z}{\partial x} = ye^{xy} + \sin y$ and $\frac{\partial z}{\partial y} = xe^{xy} + x\cos y$ so the differential dz is

$$dz = (ye^{xy} + \sin y)\ dx + (xe^{xy} + x\cos y)\ dy.$$

Example 3

If $z = x^2 + xy + y^2$, find dz. Calculating the partial derivatives, we find that $\frac{\partial z}{\partial x} = 2x + y$ and $\frac{\partial z}{\partial y} = x + 2y$ so the differential dz is

$$dz = (2x + y)\ dx + (x + 2y)\ dy.$$

Suppose that $dz = 0$. If so then

$$(2x + y)\ dx + (x + 2y)\ dy = 0$$

or

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

an ODE! Could this idea (the differential be used to find the solution of the ODE. The reader can verify that indeed the solution of the ODE is

$$x^2 + xy + y^2 + c \tag{19}$$