Math 1497 Calc 2

Now we multiply vectors

Def^n If $\bar{u} = \langle u_1, u_2 \rangle$ & $\bar{v} = \langle v_1, v_2 \rangle$

then $\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2$ (call dot product)

if $\bar{u} = \langle u_1, u_2, u_3 \rangle$ $\bar{v} = \langle v_1, v_2, v_3 \rangle$

$$\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

ex1 $\bar{u} = \langle 1, 2 \rangle$ $\bar{v} = \langle 5, -2 \rangle$

$$\bar{u} \cdot \bar{v} = (1)(5) + (2)(-2) = 5 - 4 = 1$$

ex2 $\bar{u} = \langle 3, 4, 0 \rangle$ $\bar{v} = \langle 0, 4, 5 \rangle$

$$\bar{u} \cdot \bar{v} = (3)(0) + 4(4) + 0(5) = 16$$

There are several properties of the dot product.

Let $\bar{u}, \bar{v}, \bar{w}$ be vectors & c a scalar #

(1) $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$

(2) $\bar{u} \cdot (\bar{v} + \bar{w}) = \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w}$

(3) $c(\bar{u} \cdot \bar{v}) = (c\bar{u}) \cdot \bar{v} = \bar{u} \cdot (c\bar{v})$

(4) $\bar{0} \cdot \bar{u} = 0$

(5) $\bar{u} \cdot \bar{u} = \|\bar{u}\|^2$

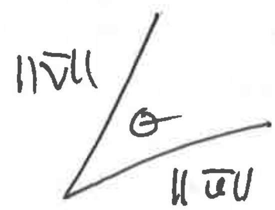
ex of (5) let $\bar{u} = \langle 1,2 \rangle$

so $\|\bar{u}\|^2 = \left(\sqrt{1^2+2^2}\right)^2 = (\sqrt{5})^2 = 5$

$\bar{u} \cdot \bar{u} = \langle 1,2 \rangle \cdot \langle 1,2 \rangle = 1^2+2^2 = 5$ ✓

There is an alternate definition for dot product

$$\bar{u} \cdot \bar{v} = \|\bar{u}\| \, \|\bar{v}\| \cos\theta$$



where θ is the angle between vectors

Important note $\boxed{\text{if } \bar{u} \perp \bar{v} \text{ then } \bar{u} \cdot \bar{v} = 0}$

A second way to multiply vectors

<u>Cross Product</u>

$$\bar{u} \times \bar{v} = \langle u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 \rangle$$

there is an easy way to obtain this

Define $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

then

$$\bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \bar{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_3 \end{vmatrix} \bar{k}$$

Ex Find $\bar{u} \times \bar{v}$ when $\bar{u} = \langle 1, 2, 3 \rangle$ $\bar{v} = \langle 1, -1, 2 \rangle$

$$\bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \bar{j} + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \bar{k}$$

$$= (4 - (-3)) \vec{i} - (2 - 3) \bar{j} + (-1 - 2) \bar{k}$$

$$= \langle 7, 1, -3 \rangle$$

We note that

$$\bar{u} \times \bar{v} \perp \bar{u}, \perp \bar{v}$$

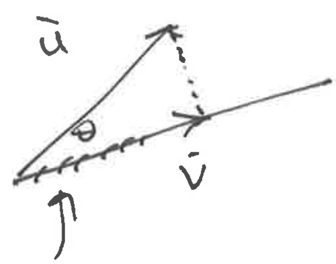
i.e. $(\bar{u} \times \bar{v}) \cdot \bar{u} = \langle 7, 1, -3 \rangle \cdot \langle 1, 2, 3 \rangle$
$= 7 + 2 - 9 = 0$ ✓

$(\bar{u} \times \bar{v}) \cdot \bar{v} = \langle 7, 1, -3 \rangle \cdot \langle 1, -1, 2 \rangle$
$= 7 - 1 - 6 = 0$ ✓

Now we deal with some application involving the two products

<u>Projections</u> - Using the dot Product

We wish to project one vector onto another

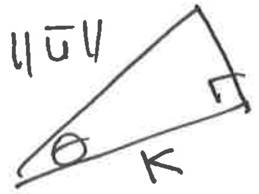


1st - create a unit vector in the direction of $\bar{v}$

this is $\dfrac{\bar{v}}{\|\bar{v}\|}$

$\text{proj}_{\bar{v}} \bar{u}$

so $\text{proj}_{\bar{v}} \bar{u} = k \dfrac{\bar{v}}{\|\bar{v}\|}$ k length of the projected vector



Now from the right angle triangle

$$\cos\theta = \frac{k}{\|\bar{u}\|} \Rightarrow k = \|\bar{u}\|\cos\theta$$

So $\quad \text{proj}_{\bar{v}}\bar{u} = \left(\dfrac{\|\bar{u}\|\cos\theta}{\|\bar{v}\|}\right)\bar{v}$

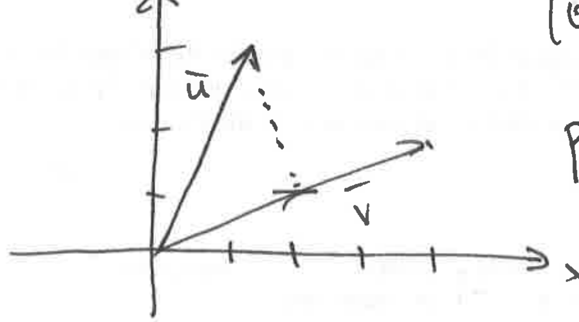
this almost looks like a dot product

$$\text{proj}_{\bar{v}}\bar{u} = \left(\frac{\|\bar{u}\|\|\bar{v}\|\cos\theta}{\|\bar{v}\|^2}\right)\bar{v}$$

$$\boxed{\text{proj}_{\bar{v}}\bar{u} = \left(\frac{\bar{u}\cdot\bar{v}}{\bar{v}\cdot\bar{v}}\right)\bar{v} \quad \text{Formula}}$$

ex If $\bar{u} = \langle 1, 3\rangle \quad \bar{v} = \langle 4, 2\rangle$

Find $\text{proj}_{\bar{v}}\bar{u}$



looks like

$\text{proj}_{\bar{v}}\bar{u}$ is

about $\frac{1}{2}\bar{v}$

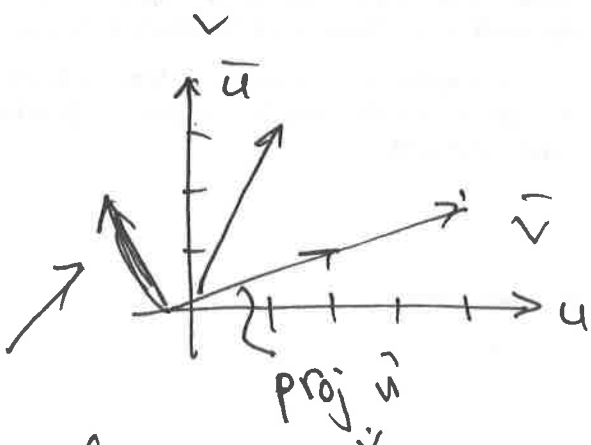
$$\bar{u} \cdot \bar{v} = \langle 1, 3\rangle \cdot \langle 4, 2\rangle$$
$$= 4 + 6 = 10$$

$$\bar{v} \cdot \bar{v} = \langle 4, 2\rangle \cdot \langle 4, 2\rangle$$
$$= 16 + 4 = 20$$

$$\text{proj}_{\bar{v}}\, \bar{u} = \left(\frac{\bar{u} \cdot \bar{v}}{\bar{v} \cdot \bar{v}}\right) \bar{v} = \frac{10}{20} \langle 4, 2\rangle$$
$$= \frac{1}{2} \langle 4, 2\rangle = \langle 2, 1\rangle$$

it is 1/2

There another vector to mention

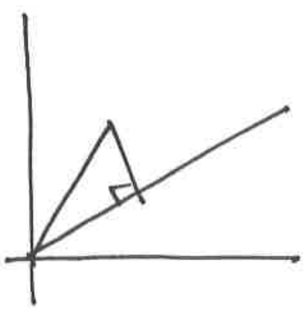


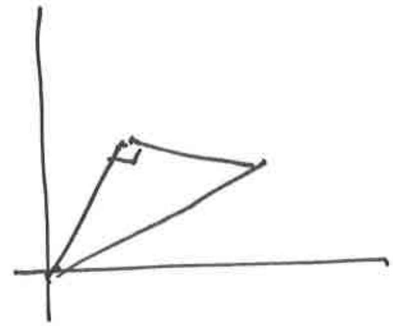
$$\text{othro}_{\bar{v}}\, \bar{u} = \bar{u} - \text{proj}_{\bar{v}}\, \bar{u}$$
$$= \langle 1, 3\rangle - \langle 2, 1\rangle$$
$$= \langle -1, 2\rangle$$

orthogonal complement

It's important to note that

$$\text{proj}_{\bar v}\bar u \neq \text{proj}_{\bar u}\bar v$$



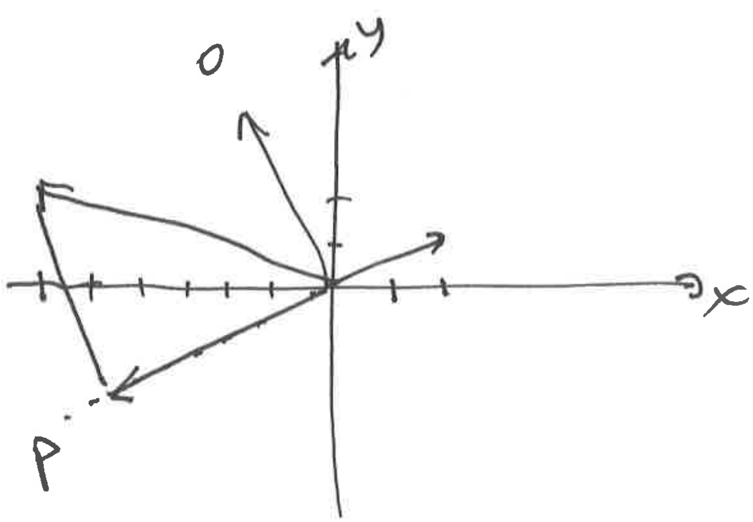


$$\text{proj}_{\bar u}\bar v = \left(\frac{\bar u\cdot\bar v}{\bar u\cdot\bar u}\right)\bar u = \frac{10}{10}\bar u = \bar u$$

Another ex:

Find $\text{proj}_{\bar v}\bar u$ if $\bar u = \langle -6, 2\rangle$

& $\text{ortho}_{\bar v}\bar u$ $\quad \bar v = \langle 2, 1\rangle$



$\bar u\cdot\bar v = -12+2 = -10$

$\bar v\cdot\bar v = 4+1 = 5$

so $\text{proj}_{\bar{v}}\bar{u} = -\frac{10}{5}\langle 2,1\rangle = -2\langle 2,1\rangle$

$= \langle -4,-2\rangle$

$\text{ortho}_{\bar{v}}\bar{u} = \bar{u} - \text{proj}_{\bar{v}}\bar{u}$

$= \langle -6,2\rangle - \langle -4,-2\rangle$

$= \langle -6+4, 2+2\rangle$

$= \langle -2,4\rangle$