Math 6345 - AODEs

Ordinary Differential Equations Review - Part 2

1 Linear Systems

A linear system of equations

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy, \tag{1}$$

can be can be written as a matrix ODE

$$\frac{d\bar{x}}{dt} = A\bar{x} \tag{2}$$

where $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\bar{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we consider solutions of the form

$$\bar{x} = \bar{c}e^{\lambda t},$$

then after substitution into (2) we obtain

$$\lambda\bar{c}\,e^{\lambda t} = A\,\bar{c}\,e^{\lambda t}$$

from which we deduce

$$(A - \lambda I)\,\bar{c} = 0. \tag{3}$$

In order to have nontrivial solutions $\bar{c}$, we require that

$$|A - \lambda I| = 0. \tag{4}$$

This is the eigenvalue-eigenvector problem. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then (4) becomes

$$\lambda^2 - TrA\lambda + DetA = 0,$$

where $TrA = a + d$ and $DetA = ad - bc$. When solving for λ we have three possible cases:

1. two distinct eigenvalues
2. two repeated eigenvalues,
3. two complex eigenvalues.

Here we consider an example of each

Example 1 If

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \bar{x} \tag{5}$$

then the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 2$.

Case 1: $\lambda = -1$

From (3) we have

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $2c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Case 2: $\lambda = 2$

From (3) we have

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $c_1 - c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution to (10) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-\mathrm{t}} + \mathrm{c}_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{2\mathrm{t}}.$$

Example 2 If

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \bar{x} \tag{6}$$

then the characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0,$$

from which we obtain the eigenvalues $\lambda = 2, 2$.

1^{st} solution: $\lambda = 2$

From (3) we have

$$\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

and our first solution is

$$\bar{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{e}^{2\mathrm{t}}.$$

2^{nd} solution: We seek a solution of the form

$$\vec{x}_2 = \vec{u}\, t\, e^{\lambda t} + \vec{v} e^{\lambda t} \tag{7}$$

Substitution into (6) leads to the following

$$(A - \lambda I)\, \vec{u} = \vec{0} \tag{8a}$$

$$(A - \lambda I)\, \vec{v} = \vec{u} \tag{8b}$$

The first we already calculated which is $\vec{c}$ above. For the second we have

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which leads to $-v_1 - v_2 = 1$ which we choose $v_1 = 0, v_2 = -1$ giving our second solution as

$$\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t\, e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \tag{9}$$

and the general solution

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] e^{2t}.$$

Example 3 If

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix} \bar{x} \tag{10}$$

then the characteristic equation is

$$\begin{vmatrix} 3 - \lambda & -2 \\ 5 & 1 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13 = 0,$$

from which we obtain the eigenvalues $\lambda = 2 \pm 3i$. In the case of complex eigenvalues where $\lambda = \alpha \pm \beta i$, the eigenvectors we be complex, say $\vec{u} = \vec{R} \pm \vec{I}\, i$ and the two independent solutions are

$$\begin{aligned} \vec{x}_1 &= \left[\vec{R} \cos \beta t - \vec{I} \sin \beta t \right] e^{\alpha t} \\ \vec{x}_2 &= \left[\vec{I} \cos \beta t + \vec{R} \sin \beta t \right] e^{\alpha t}. \end{aligned} \tag{11}$$

Substituting $\lambda = 2 + 3i$ into (3) with $a = 3, b = -2, c = 5$ and $d = 1$ gives

$$\begin{pmatrix} 1-3i & -2 \\ 5 & -1-3i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain $(1-3i)c_1 - 2c_2 = 0$ and choosing $c_1 = 2, c_2 = 1 - 3i$ or

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix} i,$$

from which we identify that

$$\vec{R} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{I} = \begin{pmatrix} 0 \\ -3 \end{pmatrix},$$

and through (11) we obtain

$$\begin{aligned} \vec{x}_1 &= \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t \right] e^{2t} \\ \vec{x}_2 &= \left[\begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin 3t \right] e^{2t}. \end{aligned} \tag{12}$$

and a linear combination gives the general solution as

$$\vec{x} = c_1 \begin{pmatrix} 2\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2\sin 3t \\ -3\cos 3t + \sin 3t \end{pmatrix} e^{2t} \tag{13}$$