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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

The curve **C**, with equation $y = x^2 \ln x$, $x > 0$, has a stationary point P. Find, in terms of e, the coordinates of P. (7)

Solution:

$y = x^2 \ln x, x > 0$

Differentiate as a product:

$$\frac{dy}{dx} = x^2 \times \frac{1}{x} + 2x \ln x = x + 2x \ln x = x(1 + 2 \ln x)$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad 1 + 2 \ln x = 0 \text{ as } x > 0$$

$$\Rightarrow \quad 2 \ln x = -1$$

$$\Rightarrow \quad \ln x = -\frac{1}{2}$$

$$\Rightarrow \quad x = e^{-\frac{1}{2}}$$

Substituting $x = e^{-\frac{1}{2}}$, in $y = x^2 \ln x$

$$\Rightarrow \quad y = \left(e^{-\frac{1}{2}}\right)^2 \ln e^{-\frac{1}{2}} = -\frac{1}{2}e^{-1}$$

So coordinates are $\left(e^{-\frac{1}{2}}, -\frac{1}{2}e^{-1}\right)$

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Exercise A, Question 2

Question:

$f(x) = e^{2x-1}, x \geq 0$

The curve **C** with equation $y = f(x)$ meets the y-axis at P.

The tangent to **C** at P crosses the x-axis at Q.

(a) Find, to 3 decimal places, the area of triangle POQ, where O is the origin. (5)

The line $y = 2$ intersects **C** at the point R.

(b) Find the exact value of the x-coordinate of R. (3)

Solution:

(a) **C** meets y-axis where $x = 0$

$\Rightarrow \quad y = e^{-1}$

Find gradient of curve at P.

$\frac{dy}{dx} = 2e^{2x-1}$

At $x = 0$, $\frac{dy}{dx} = 2e^{-1}$

Equation of tangent is $y - e^{-1} = 2e^{-1}x$

This meets x-axis at Q, where $y = 0$

$\Rightarrow \quad Q \equiv \left(-\frac{1}{2}, 0 \right)$

Area of $\triangle$ POQ $= \frac{1}{2} \times \frac{1}{2} \times e^{-1} = \frac{1}{4}e^{-1} = 0.092$

(b) At R, $y = 2 \quad \Rightarrow \quad 2 = e^{2x-1}$

$\Rightarrow \quad 2x - 1 = \ln 2$

$\Rightarrow \quad 2x = 1 + \ln 2$

$\Rightarrow \quad x = \frac{1}{2}(1 + \ln 2)$

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Exercise A, Question 3

Question:

$$f(x) = \frac{3x}{x+1} - \frac{x+7}{x^2-1}, x > 1$$

(a) Show that $f(x) = 3 - \frac{4}{x-1}, x > 1$. (5)

(b) Find $f^{-1}(x)$. (4)

(c) Write down the domain of $f^{-1}(x)$. (1)

Solution:

(a) $\frac{3x}{x+1} - \frac{x+7}{(x+1)(x-1)}, x > 1$

$$\equiv \frac{3x(x-1) - (x+7)}{(x+1)(x-1)}$$

$$\equiv \frac{3x^2 - 4x - 7}{(x+1)(x-1)}$$

$$\equiv \frac{(3x-7)(x+1)}{(x+1)(x-1)}$$

$$\equiv \frac{3x-7}{x-1}$$

$$\equiv \frac{3(x-1) - 4}{x-1}$$

$$\equiv 3 - \frac{4}{x-1}$$

(b) Let $y = 3 - \frac{4}{x-1}$

$$\Rightarrow \frac{4}{x-1} = 3 - y$$

$$\Rightarrow \frac{x-1}{4} = \frac{1}{3-y}$$

$$\Rightarrow x - 1 = \frac{4}{3-y}$$

$\Rightarrow \quad x = 1 + \frac{4}{3 - y}$ or $\frac{7 - y}{3 - y}$

So $f^{-1}(x) = 1 + \frac{4}{3 - x}$ or $\frac{7 - x}{3 - x}$

(c) Domain of $f^{-1}(x)$ is the range of $f(x)$.

$x > 1 \quad \Rightarrow \quad \frac{4}{x - 1} > 0 \quad \Rightarrow \quad f(x) = 3 - \frac{4}{x - 1} < 3$

So the domain of $f^{-1}(x)$ is $x < 3$

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Exercise A, Question 4

Question:

(a) Sketch, on the same set of axes, for $x > 0$, the graphs of

$y = -1 + \ln 3x$ and $y = \frac{1}{x}$ (2)

The curves intersect at the point P whose x-coordinate is p.
Show that

(b) p satisfies the equation
$p \ln 3p - p - 1 = 0$ (1)

(c) $1 < p < 2$ (2)
The iterative formula

$x_{n+1} = \frac{1}{3} e^{\left(1 + \frac{1}{x_n}\right)}, x_0 = 2$

is used to find an approximation for p.

(d) Write down the values of x_1, x_2, x_3 and x_4 giving your answers to 4 significant figures. (3)

(e) Prove that $p = 1.66$ correct to 3 significant figures. (2)

Solution:

(a)

(b) At P, $-1 + \ln 3p = \frac{1}{p}$

$\Rightarrow \quad -p + p \ln 3p = 1$

$\Rightarrow \quad p \ln 3p - p - 1 = 0$

(c) Let $f(p) \equiv p \ln 3p - p - 1$
$f(1) = \ln 3 - 2 = -0.901...$
$f(2) = 2 \ln 6 - 3 = +0.5835...$
Sign change implies root between 1 and 2, so $1 < p < 2$.

(d) $x_{n+1} = \frac{1}{3} e^{\left(1 + \frac{1}{x_n}\right)}, x_0 = 2$

$x_1 = \frac{1}{3}e^{\frac{3}{2}} = 1.494$ (4 s.f.)

$x_2 = 1.770$ (4 s.f.)

$x_3 = 1.594$ (4 s.f.)

$x_4 = 1.697$ (4 s.f.)

(e) $f(1.665) = +0.013$

$f(1.655) = -0.003$

$\Rightarrow$ root between 1.655 and 1.665

So $p = 1.66$ (3 s.f.)

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Exercise A, Question 5

Question:

The curve $\mathbf{C}_1$ has equation

$y = \cos 2x - 2 \sin^2 x$

The curve $\mathbf{C}_2$ has equation

$y = \sin 2x$

(a) Show that the x-coordinates of the points of intersection of $\mathbf{C}_1$ and $\mathbf{C}_2$ satisfy the equation

$2 \cos 2x - \sin 2x = 1$ (3)

(b) Express $2 \cos 2x - \sin 2x$ in the form $R \cos(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving the exact value of R and giving α in radians to 3 decimal places. (4)

(c) Find the x-coordinates of the points of intersection of $\mathbf{C}_1$ and $\mathbf{C}_2$ in the interval $0 \le x < \pi$, giving your answers in radians to 2 decimal places. (5)

Solution:

(a) Where $\mathbf{C}_1$ and $\mathbf{C}_2$ meet

$\cos 2x - 2 \sin^2 x = \sin 2x$

Using $\cos 2x \equiv 1 - 2 \sin^2 x \Rightarrow -2 \sin^2 x \equiv \cos 2x - 1$

So $\cos 2x + (\cos 2x - 1) = \sin 2x$

$\Rightarrow 2 \cos 2x - \sin 2x = 1$

(b) Let $2 \cos 2x - \sin 2x \equiv R \cos(2x + \alpha)$

$\equiv R \cos 2x \cos \alpha - R \sin 2x \sin \alpha$

Compare: $R \cos \alpha = 2$, $R \sin \alpha = 1$

Divide: $\tan \alpha = \frac{1}{2} \Rightarrow \alpha = 0.464$ (3 d.p.)

Square and add: $R^2(\cos^2\alpha + \sin^2\alpha) = 2^2 + 1^2 = 5$

$\Rightarrow R = \sqrt{5}$

So $2 \cos 2x - \sin 2x \equiv \sqrt{5} \cos(2x + 0.464)$

(c) $2 \cos 2x - \sin 2x = 1$

$\Rightarrow \quad \sqrt{5} \cos(2x + 0.464) = 1$

$\Rightarrow \quad \cos\left(2x + 0.464\right) = \frac{1}{\sqrt{5}}$

$\Rightarrow \quad 2x + 0.464 = 1.107, \quad 5.176 \qquad 0.464 \leq 2x + 0.464 < 6.747$

$\Rightarrow \quad 2x = 0.643, 4.712$

$\Rightarrow \quad x = 0.32, 2.36$ (2 d.p.)

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Exercise A, Question 6

Question:

(a) Given that $y = \ln \sec x$, $-\frac{\pi}{2} < x \leq 0$, use the substitution $u = \sec x$, or otherwise, to show that $\frac{dy}{dx} = \tan x$. (3)

The curve **C** has equation $y = \tan x + \ln \sec x$, $-\frac{\pi}{2} < x \leq 0$.
At the point P on **C**, whose x-coordinate is p, the gradient is 3.

(b) Show that $\tan p = -2$. (6)

(c) Find the exact value of sec p, showing your working clearly. (2)

(d) Find the y-coordinate of P, in the form $a + k \ln b$, where a, k and b are rational numbers. (2)

Solution:

(a) $y = \ln \sec x$

Let $u = \sec x \Rightarrow \frac{du}{dx} = \sec x \tan x$

so $y = \ln u \Rightarrow \frac{dy}{du} = \frac{1}{u}$

Using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{u} \times \sec x \tan x = \frac{1}{\sec x} \sec x \tan x = \tan x$

(b) $\frac{dy}{dx} = \sec^2 x + \tan x$

So $\sec^2 p + \tan p = 3$

$\Rightarrow \quad 1 + \tan^2 p + \tan p = 3$

$\Rightarrow \quad \tan^2 p + \tan p - 2 = 0$

$\Rightarrow \quad (\tan p - 1)(\tan p + 2) = 0$

As $-\frac{\pi}{2} < x \leq 0$ (4th quadrant), tan p is negative

So $\tan p = -2$

(c) $\sec^2 p = 1 + \tan^2 p = 5 \quad \Rightarrow \quad \sec\ p = +\sqrt{5}$ (4th quadrant)

(d) $y = \ln\sqrt{5} + (-2) = -2 + \frac{1}{2}\ln 5$

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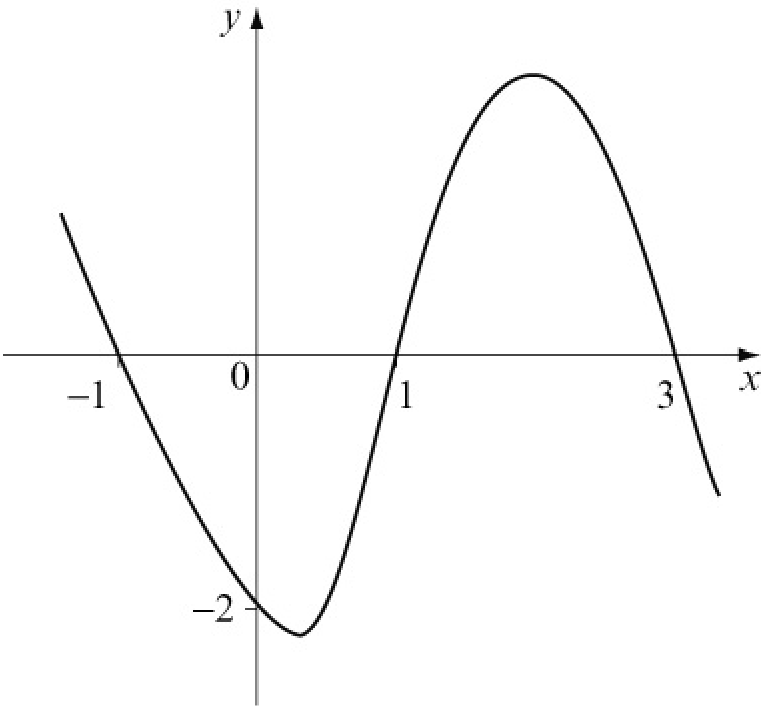
Exercise A, Question 7

Question:

The diagram shows a sketch of part of the curve with equation $y = f(x)$. The curve has no further turning points.



On separate diagrams show a sketch of the curve with equation

(a) $y = 2f(-x)$ (3)

(b) $y = |f(2x)|$ (3)
In each case show the coordinates of points in which the curve meets the coordinate axes.
The function g is given by
$g : x \rightarrow |x+1| - k, x \in \mathbb{R}, k > 1$

(c) Sketch the graph of g, showing, in terms of k, the y-coordinate of the point of intersection of the graph with the y-axis. (3)
Find, in terms of k,

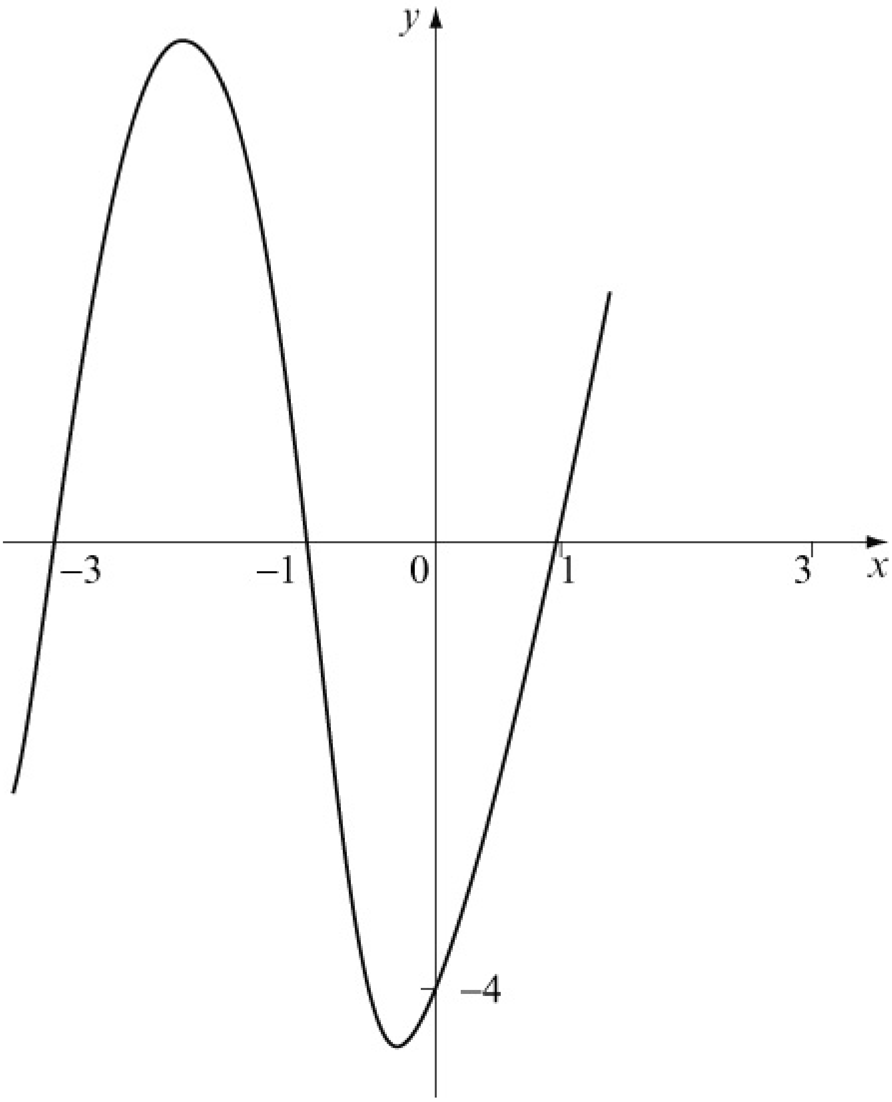
(d) the range of g(x) (1)

(e) gf(0) (2)

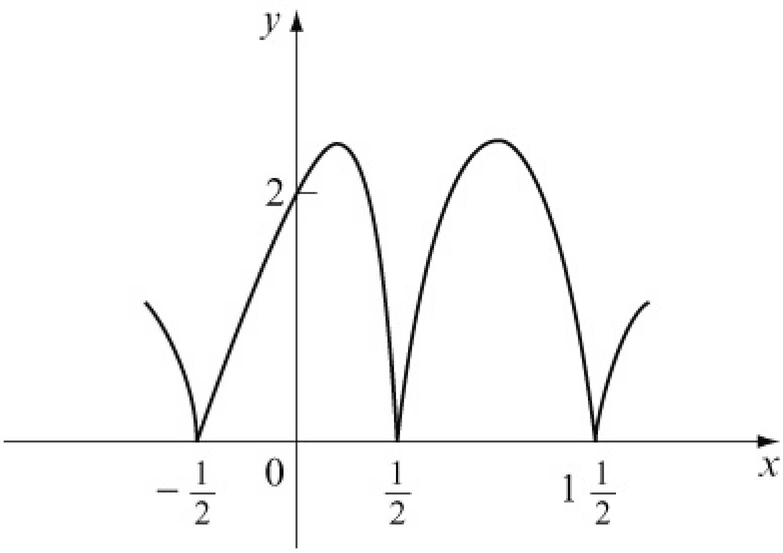
(f) the solution of $g(x) = x$ (3)

Solution:

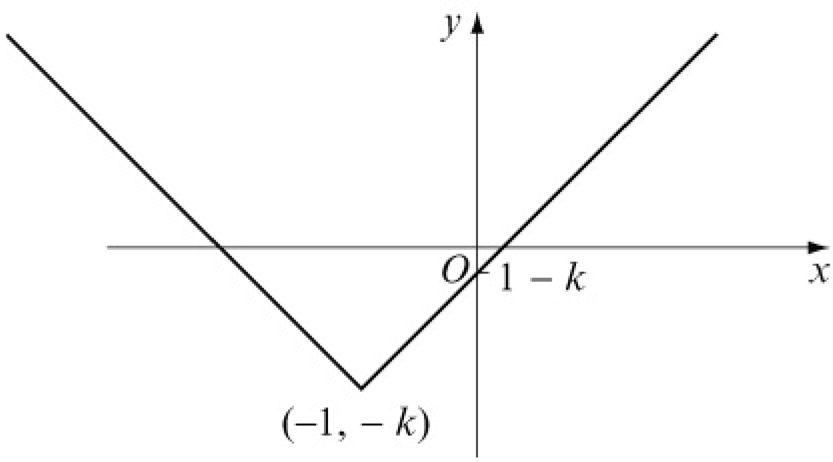
(a)



(b)



(c)



(d) $g(x) \geq -k$

(e) $gf(0) = g(-2) = |-1| - k = 1 - k$

(f) $y = x$ meets $y = |x + 1| - k$
where $x = -(x + 1) - k$

$\Rightarrow \quad 2x = -(1 + k)$

$\Rightarrow \quad x = -\frac{1 + k}{2}$