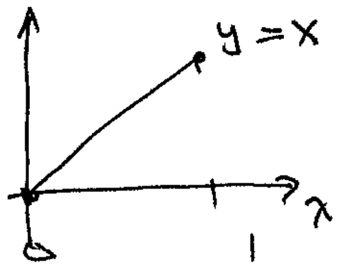
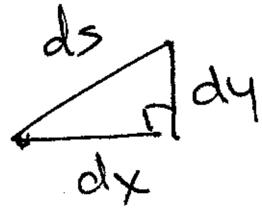
Math 1496 - Calc I

Arc Length

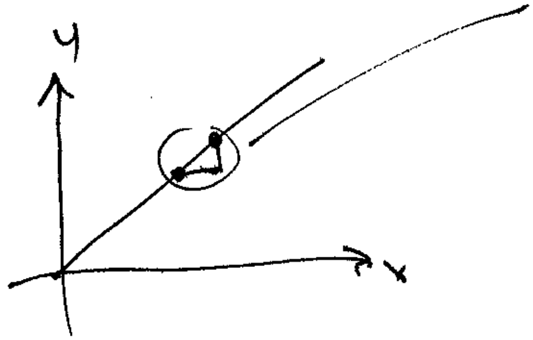


If we consider the line $y = x$ on $[0,1]$ and we to find the length of the curve then it's just the distance from $(0,0) \Rightarrow (1,1)$ $\quad s = \sqrt{1^2 + 1^2} = \sqrt{2}$

If the "curve" is not straight it's a harder problem. Let use split up curve into small segments



so $(ds)^2 = (dx)^2 + (dy)^2$



$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2}$$

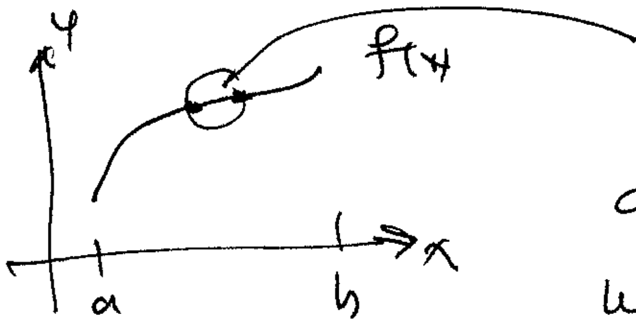
$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

so here $y = x$ $\quad \frac{dy}{dx} = 1$

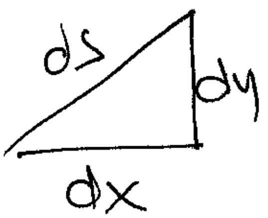
$ds = \sqrt{2}\, dx$ then add up segments

$$s = \int_0^1 \sqrt{2}\, dx = \sqrt{2}\, x \Big|_0^1 = \sqrt{2}$$

we can do this in general



approx. the small arc
with a straight line



so $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

then add $S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ arc length formula

<u>Ex1</u> Find the length of $y = x^{3/2}$ from $x = 0 \to 4$

so $y' = \frac{3}{2}x^{1/2}$ $\quad S = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx$

let $u = 1+\frac{9}{4}x$ $\quad du = \frac{9}{4}dx$ $\quad x=0\ u=1$ $\quad x=4\ u=10$

$\frac{4}{9}\int_1^{10}\sqrt{u}\,du = \frac{4}{9}\cdot\frac{2u^{3/2}}{3}\Big|_1^{10} = \frac{8}{27}\left(10\sqrt{10}-1\right)$

Ex2 Find the arc length of

$f(x) = \frac{x^2}{8} - \ln x$ on $[1, 2]$

Sol^n $f' = \frac{x}{4} - \frac{1}{x}$ $\quad 1 + f'^2 = 1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2$

$= 1 + \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}$

$= \frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2} = \left(\frac{x}{4} + \frac{1}{x}\right)^2$

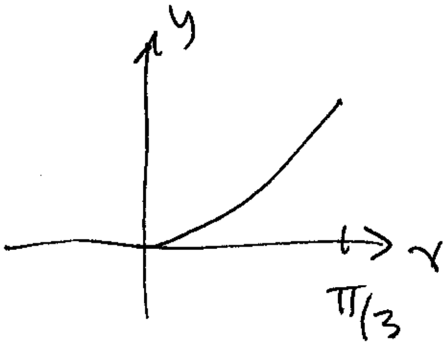
so $\sqrt{1 + f'^2} = \sqrt{\left(\frac{x}{4} + \frac{1}{x}\right)^2} = \frac{x}{4} + \frac{1}{x}$ $\quad (x \geq 1)$

$S = \int_1^2 \left(\frac{x}{4} + \frac{1}{x}\right) dx$

$= \frac{x^2}{8} + \ln|x| \Big|_1^2 = \frac{4}{8} + \ln 2 - \frac{1}{8} = \frac{3}{8} + \ln 2$

Ex 3 Find the length of the curve

$y = \ln \sec x$ from $x = 0 \to \pi/3$



$$y' = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$$

$$\sqrt{1+y'^2} = \sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \overset{\sec}{\cancel{\tan}}\, x$$

$$S = \int_0^{\pi/3} \sec x \, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/3}$$

$$= \ln|\sec \pi/3 + \tan \pi/3| - \ln|\sec 0 + \tan 0|$$

$$\sec \pi/3 = \frac{1}{\cos \pi/3} = \frac{1}{1/2} = 2 \qquad \tan \pi/3 = \sqrt{3}$$

$$= \ln(2+\sqrt{3}) - \ln|1+0|$$

$$= \ln(2+\sqrt{3})$$