Sets with large additive energy and symmetric sets *

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Annotation.

We show that for any set A in a finite Abelian group $\mathbf{G}$ that has at least $c|A|^3$ solutions to $a_1+a_2=a_3+a_4$, $a_i \in A$ there exist sets $A' \subseteq A$ and $\Lambda \subseteq \mathbf{G}$, $\Lambda=\{\lambda_1,\dots,\lambda_t\}$, $t \ll c^{-1}\log|A|$ such that A' is contained in $\left\{\sum_{j=1}^t \varepsilon_j\lambda_j \mid \varepsilon_j \in \{0,-1,1\}\right\}$ and A' has $\gg c|A|^3$ solutions to $a'_1+a'_2=a'_3+a'_4$, $a'_i \in A'$. We also study so–called symmetric sets or, in other words, sets of large values of convolution.

1 Introduction

Let $\mathbf{G}$ be a finite Abelian group. For sets $A, B \subseteq \mathbf{G}$ let $E(A,B)$ denote their *additive energy*

$$E(A,B) := |\{a_1+b_1=a_2+b_2 \ : \ a_1,a_2 \in A,\ b_1,b_2 \in B\}| \ .$$

We write $E(A)$ in place of $E(A,A)$. For a set $\Lambda=\{\lambda_1,\dots,\lambda_t\}$ in $\mathbf{G}$ let $\mathrm{Span}\,(\Lambda)$ denote the set $\left\{\sum_{j=1}^t \varepsilon_j\lambda_j \ \Big| \ \varepsilon_j \in \{0,-1,1\}\right\}$. A set A satisfying $E(A) \geq c|A|^3$ for some constant c, is called a set of *large additive energy*. Sets of large additive energy are very important in additive combinatorics [7]. In [5] T. Sanders obtained the following result about such sets.

Theorem 1.1 *Let $\mathbf{G}$ be a finite Abelian group, $A \subseteq \mathbf{G}$ be a set, and $c \in (0,1]$. Suppose $E(A) \geq c|A|^3$; then there exist sets $A_1 \subseteq A$ and $\Lambda \subseteq \mathbf{G}$, such that $|\Lambda| \ll c^{-1}\log|A|$, $A_1 \subseteq \mathrm{Span}\,(\Lambda)$ and $|A_1| \geq 2^{-2}c^{1/2}|A|$.*

A slightly weaker version of the theorem above (with with $c^{1/2+\varepsilon}$ instead of $c^{1/2}$) was obtained in [10] using so–called (C,β)–connected sets. In [5] Sanders also considered a stronger restriction on the set A, namely $|A+A| \leq c^{-1}|A|$, and obtained an improvement of theorem 1.1 in this setting (see theorem 1.2 below). He also found an interesting generalization of theorem 1.1 for the case of two different sets A and B.

Theorem 1.2 *Let $\mathbf{G}$ be a finite Abelian group, $A,B \subseteq \mathbf{G}$ be two sets, and $c \in (0,1]$. Suppose $|A+B| \leq c^{-1}|A|$; then there is a set $\Lambda \subseteq \mathbf{G}$, $|\Lambda| \ll c^{-1}\log|A|$ such that $B \subseteq \mathrm{Span}\,(\Lambda)$.*

Applications of the theorem above can be found in [6]. In the current paper we obtain an extension of Theorem 1.1 for the case of two different sets A and B. We also obtain a refinement of the theorem in the case $A=B$.

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Theorem 1.3 *Let* $\mathbf{G}$ *be a finite Abelian group,* $A, B \subseteq \mathbf{G}$ *be two sets, and* $c \in (0,1]$. *Suppose* $E(A,B) \geq c|A||B|^2$; *then there exist sets* $B_1 \subseteq B$ *and* $\Lambda \subseteq \mathbf{G}$, $|\Lambda| \ll c^{-1}\log|A|$ *such that* $B_1 \subseteq \mathrm{Span}\,(\Lambda)$ *and*

$$E(A,B_1) \geq 2^{-5}E(A,B). \qquad (1)$$

In particular, $|B_1| \geq 2^{-3}c^{1/2}|B|$.

Note 1.4 The result above yields an improvement of Theorem 1.1. Indeed, suppose that in the previous theorem we have $B = A$. Let $A_1 = B_1$. Then $E(A,A_1) \geq 2^{-5}E(A)$. Using the Cauchy–Schwartz inequality, we get

$$E(A_1) \geq 2^{-10}E(A)\,.$$

Therefore $|A_1| \geq 2^{-4}c^{1/3}|A|$ and the exponent is sharp (see below).

In what follows we give three proofs of theorem 1.3. In section 2 we give the first (Fourier analytic) proof. In section 3 we establish a result on large values of convolution of two sets (theorem 3.1). We then give the second proof of theorem 1.3. Our proof relies on an idea of Sanders [6]. We do not use the Fourier transform and get slightly weaker bounds. Later we generalize theorem 3.1, and rely on that generalization to obtain the last proof of (a small refinement of) theorem 1.3. Again, we do not use the Fourier method.

Our results concerning the structure of sets of large values of convolution are of independent interest. Our results on sets with large additive energy are considerably weaker than the implications of the polynomial Freiman–Ruzsa conjecture [2].

We conclude with few comments regarding the notation used in this paper. For a positive integer n, we set $[n] = \{1,\dots,n\}$. All logarithms are base 2. Signs $\ll$ and $\gg$ are the usual Vinogradov's symbols. Finally, with a slight abuse of notation we use the same letter to denote a set $S \subseteq \mathbf{G}$ and its characteristic function $S : \mathbf{G} \to \{0,1\}$.

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2 Proof of the main result

Let $\mathbf{G}$ be a finite Abelian group, $N = |\mathbf{G}|$. It is well–known [4] that the dual group $\widehat{\mathbf{G}}$ is isomorphic to $\mathbf{G}$. Let f be a function from $\mathbf{G}$ to $\mathbb{C}$. We denote the Fourier transform of f by $\widehat{f}$,

$$\widehat{f}(\xi) = \sum_{x\in\mathbf{G}} f(x)e(-\xi\cdot x)\,, \qquad (2)$$

where $e(x) = e^{2\pi i x}$. We rely on the following basic identities

$$\sum_{x\in\mathbf{G}} |f(x)|^2 = \frac{1}{N}\sum_{\xi\in\widehat{\mathbf{G}}} \left|\widehat{f}(\xi)\right|^2\,. \qquad (3)$$

$$\sum_{y\in\mathbf{G}} \left|\sum_{x\in\mathbf{G}} f(x)g(y-x)\right|^2 = \frac{1}{N}\sum_{\xi\in\widehat{\mathbf{G}}} \left|\widehat{f}(\xi)\right|^2 |\widehat{g}(\xi)|^2\,. \qquad (4)$$

If

$$(f*g)(x) := \sum_{y\in\mathbf{G}} f(y)g(x-y)$$

then

$$\widehat{f*g} = \widehat{f}\widehat{g} \quad \text{and} \quad (\widehat{fg})(x) = \frac{1}{N}(\widehat{f}*\widehat{g})(x)\,. \tag{5}$$

Using (4), we can express additive energy in terms of the Fourier transform

$$E(A,B) = \frac{1}{N}\sum_{\xi}\left|\widehat{A}(\xi)\right|^2\left|\widehat{B}(\xi)\right|^2\,.$$

Our first proof of theorem 1.3 relies on the following lemma of T. Sanders [5]. (Similar results were obtained by J. Bourgain [1] and by the first author [10].) Recall that a set $\Lambda = \{\lambda_1,\dots,\lambda_t\}$ in a finite Abelian group $\mathbf{G}$ is called *dissociated* if any identity of the form $\sum_{j=1}^t \varepsilon_j\lambda_j = 0$, where $\varepsilon_j \in \{0,-1,1\}$ yields $\varepsilon_j = 0$, $j\in[t]$.

Lemma 2.1 *Let $\mathbf{G}$ be a finite Abelian group, $Q\subseteq\mathbf{G}$ be a set, l be a positive integer. There is a set $Q_1\subseteq Q$ such that all dissociated subsets of Q_1 have size at most l and for all $p\geq 2$ the following holds*

$$\left(\frac{1}{N}\sum_{\xi}\left|\widehat{Q}(\xi)-\widehat{Q}_1(\xi)\right|^p\right)^{1/p} \ll \sqrt{p/l}\cdot|Q|\,. \tag{6}$$

Proof of Theorem 1.3 Apply Lemma 2.1 to the set B with parameters $p = 2+\log|A|$ and $l = \eta^{-1}c^{-1}\log|A|$, where $\eta\in(0,1]$ is an appropriate constant that we fix later. Write $\varepsilon(x) = B(x) - B_1(x)$, where $B_1\subseteq B$ is such that all dissociated subsets of B_1 have size at most l. We have

$$N\cdot E(A,B) = \sum_{\xi}\left|\widehat{A}(\xi)\right|^2\left|\widehat{B}(\xi)\right|^2 = \sum_{\xi}\left|\widehat{A}(\xi)\right|^2\left|\widehat{B}_1(\xi)\right|^2 +$$

$$+\left(\sum_{\xi}\left|\widehat{A}(\xi)\right|^2\overline{\widehat{B}_1(\xi)}\widehat{\varepsilon}(\xi) + \sum_{\xi}\left|\widehat{A}(\xi)\right|^2\widehat{B}_1(\xi)\overline{\widehat{\varepsilon}(\xi)}\right) + \sum_{\xi}\left|\widehat{A}(\xi)\right|^2|\widehat{\varepsilon}(\xi)|^2 =$$

$$= \sigma_0+\sigma_1+\sigma_2\,.$$

By the Hölder inequality, identity (3), and our choice of parameters, we have

$$\sigma_2 \leq \left(\sum_{\xi}|\widehat{\varepsilon}(\xi)|^{2p}\right)^{1/p}\cdot\left(\sum_{\xi}|\widehat{A}(\xi)|^{\frac{2p}{p-1}}\right)^{1-1/p} \ll \frac{p}{l}|B|^2|A||A|^{1/p}N \leq 2^{-1}c|A||B|^2N\,. \tag{7}$$

Hence either σ_0 or σ_1 is at least $2^{-2}c|A||B|^2N$. In the first case we are done. In the second case an application of the Cauchy–Schwartz inequality yields

$$2^{-6}N^2E^2(A,B) \leq N\cdot E(A,B_1)\cdot\sigma_2\,.$$

Combining the inequality above with (7) we get (1). This completes the proof of Theorem 1.3.

For a set $Q\subseteq\mathbf{G}$ let $\dim(Q)$ denote the size of the largest dissociated subset of Q. Clearly, for any set $Q\subseteq\mathbf{G}$ there is a dissociated set $\Lambda\subseteq Q$ such that $|\Lambda| = \dim(Q)$ and $Q\subseteq \mathrm{Span}\,(\Lambda)$. Thus, all theorems above can be viewed as results concerning the dimension of certain subsets of sets with large additive energy.

By note 1.4, theorem 1.3 yields an improvement of theorem 1.1. Nevertheless the method from [10] is surprisingly sharp. Indeed, the argument there proceeds in two steps. Firstly, one

finds a (C,β)–connected subset of A of size approximately $c^{1/2}|A|$ (see [10] for appropriate definitions). Secondly, one proves that any connected set belongs to a span of a set of size $O(c^{-1}\log|A|)$. It is not hard to verify that the bound used on the second step is sharp. The argument used on the first step also cannot be improved. We are grateful to T. Sanders for pointing us to the following example (see also [9], theorem 4.1).

Let $\mathbf{G}=(\mathbb{Z}/2\mathbb{Z})^n$. For a linear subspace H of $\mathbf{G}$, let $\mathrm{supp}\,(H)=\{i\in[n]\mid\exists x\in H, x_i\neq 0\}$ denote the support of H. Set $A=\bigcup_{i=1}^t H_i$ to be a union of t linear subspaces $\{H_i\}_{i\in[t]}$ that have the same size $h\gg t$ and disjoint supports. It is not hard to show (see [9] for details) that $th^3\leq E(A)\ll th^3=(th)^3/t^2$ and any connected subset of A has cardinality $O(h)=O((th)/(t^2)^{1/2})$.

Observe that the exponent of c in note 1.4 is the best possible. Indeed, set $\mathbf{G}=(\mathbb{Z}/2\mathbb{Z})^n$, and set $A=H\bigsqcup\Lambda$, where H is a linear subspace of size approximately $c^{1/3}|\Lambda|$, and Λ is a dissociated set. Now $E(A)\gg c|A|^3$ and for every set $A_1\subseteq A$ such that $\dim(A_1)\ll c^{-1}\log|A|$, $|A_1|\ll c^{1/3}|A|$ necessarily holds.

3 Large values of convolution

The following theorem bounds the dimension of symmetric sets [7], or in other words, sets of large values of convolution.

Theorem 3.1 *Let $\mathbf{G}$ be a finite Abelian group, $A,B\subseteq\mathbf{G}$ be two sets. Let $\sigma\geq 1$ be a positive real number. Finally, let*

$$S=\{x\in\mathbf{G}\ :\ (A*(-B))(x)\geq\sigma\}\,.$$

Then

$$\dim(S)\ll\max\{|A|,|B|\}\cdot\sigma^{-1}\cdot\log(\min\{|A|,|B|\})\,. \tag{8}$$

Proof. Assume $|B|\leq|A|$. Let Λ be the largest dissociated subset of S, $|\Lambda|=\dim(S)$. Consider a simple bipartite graph $G=(V,E)$ with parts A and B and colors $\lambda\in\Lambda$ on edges. A vertex $a\in A$ is connected to a vertex $b\in B$ by an edge colored $\lambda\in\Lambda$ if and only if $a-b=\lambda$. Note that all edges incident to a certain vertex have different colors. Also note that $|E|\geq\sigma|\Lambda|$.

For an edge $e\in E$, let $\mathrm{col(e)}\in\Lambda$ denote its color. Let $C=\{e_1,\dots,e_k\}\subseteq E^k$ be an arbitrary k-long cycle in G. We have

$$\sum_{i\in[k]}(-1)^i\mathrm{col(e_i)}=0. \tag{9}$$

Let $e_i, i\in[k]$ be an arbitrary edge of C. We say that e_i is a *special* edge, if for all $j\in[k]$ such that $i\neq j$ we have $\mathrm{col(e_i)}\neq\mathrm{col(e_j)}$. We say that C is a *special* cycle, if one (or more) of its edges are special. Observe that if C is a special cycle; then (9) gives a non-trivial dependence between the elements of Λ. Thus to prove theorem 3.1 it suffices to establish the following

Lemma 3.2 *Suppose in the setting above we have $|\Lambda|>16|A|\sigma^{-1}\log|B|$; then there is a special cycle of length at most $4\log|B|$ in G.*

Our proof of lemma 3.2 relies on the following lemma of Erdös [3][p. 74, lemma 7.1].

Lemma 3.3 *Let $\Gamma=(V,E)$ be a finite simple graph, d be a positive integer, and $|E|>(d-1)|V|$. Then Γ has a subgraph of minimum degree at least d.*

Proof of Lemma 3.2 We apply the Erdös' lemma to the graph G to obtain a sub-graph G'. Note that the degree of every vertex of G' is at least $d = 2^{-2}\sigma|\Lambda||A|^{-1}$. By the assumption of the lemma we have $d > 4\log|B|$. To find a special cycle in G', we pick an arbitrary node $v_0 \in G' \cap A$ and start carefully constructing a binary sub-tree of G' rooted at v_0.

We assign every node in our sub-tree (other v_0) a color, which is the color of the edge that comes from its parent. We gradually extend the depth of our binary tree trying to keep the following invariant satisfied: "*For every node v in the tree: The color of v is different from the colors of all ancestors and siblings of ancestors of v.*"

Below is the pseudo-code of our tree construction procedure. Here $Tree$ denotes the set of nodes that are already in the tree (initially $Tree = \{v_0\}$). Further, for any $v \in Tree$, $F(v)$ denotes the set of colors that includes the color of v as well as the colors of all ancestors and siblings of ancestors of v. We repeat the following procedure incrementing the value of i starting with $i = 0$:

1. **For** every node v at depth i **Do**
2. **Begin**
3. Pick v_1 and v_2 to be two children of v such that
4. $\mathrm{col}(\{v, v_1\}) \notin F(v)$ and $\mathrm{col}(\{v, v_2\}) \notin F(v)$
5. (If no two such children exist **Abort.**)
6. **If** $(v_1 \in Tree)$ or $(v_2 \in Tree)$ **Then Abort.**
7. **Else**
8. **Begin**
9. $Tree := Tree \cup \{v_1, v_2\}$
10. $F(v_1) := F(v) \cup \{\mathrm{col}(\{\mathrm{v}, \mathrm{v}_1\}), \mathrm{col}(\{\mathrm{v}, \mathrm{v}_2\})\}$
11. $F(v_2) := F(v) \cup \{\mathrm{col}(\{\mathrm{v}, \mathrm{v}_1\}), \mathrm{col}(\{\mathrm{v}, \mathrm{v}_2\})\}$
12. **End**
13. **End**

The lower bound on d that we have implies that while we construct the first $2\log|B|$ levels of our tree we will always be able to find two edges emanating from a node that have suitable colors. (In other words, no abort on line 5 of the pseudo-code will occur while $i \le 2\log|B|$.) Now observe that all odd depth nodes in the tree we construct belong to the set B. Therefore our tree construction algorithm will necessarily discover some cycle C and abort (at line 6 of the pseudo-code) at some depth $i \le 2\log|B|$. We claim that C is special cycle. Indeed, let v_* be the node of the smallest depth in C. It not hard to check that both edges incident to v_* in C are special. This concludes the proof of lemma and theorem 3.1.

Note 3.4 An appropriate version of Chang's theorem (see [6] or [11]) implies a bound for $\dim(S)$ that is weaker than (8). Specifically, it yields

$$\dim(S) \ll |A||B| \cdot \sigma^{-2} \cdot \log(\min\{|A|, |B|\}).$$

Note 3.5 Inequality (8) is the best possible. To see this let $\mathbf{G} = (\mathbb{Z}/2\mathbb{Z})^n$. Let B be a subspace, and let $A = B \dot{+} \Lambda$, where Λ is a dissociated set. Now $\sigma \sim |B|$ and $\dim(S) \sim |\Lambda| + \dim(B)$. One can get a similar example with $E(B) = o(|B|^3)$, setting $A = H \dot{+} \Lambda_1 \dot{+} \Lambda_2$ and $B = H \dot{+} \Lambda_1$, where Λ_1, Λ_2 are dissociated sets and H is a subspace (note that by construction sets A and B are connected).

We now proceed to the second

Proof of Theorem 1.3. Let

$$S_j = \{x \in \mathbf{G} \ : \ c2^{j-2}|B| \le (A*B)(x) < c2^{j-1}|B|\}\,, j \in [s]\,, \quad s \ll \log(1/c)\,.$$

By assumption $E(A,B) \ge c|A||B|^2$. Hence

$$\sum_{j=1}^{s}\sum_{x\in S_j}(A*B)^2(x) \ge 2^{-1}c|A||B|^2\,.$$

Put $c_j = \frac{1}{|A||B|^2}\sum_{x\in S_j}(A*B)^2(x)$. Then

$$2^{-1}c \le \sum_{j=1}^{s} c_j \le c \tag{10}$$

and by definition of S_j, we have $c_j \le c2^{j-1}$. Fix $j \in [s]$ such that $c_j \ge (2s)^{-1}c > 0$. We have

$$\sum_{x\in S_j}(A*B)(x) = \sum_x B(x)(S_j*(-A))(x) \ge 2^{-j+1}\frac{c_j}{c}|A||B|\,. \tag{11}$$

Let

$$B_1 = \{x \in B \ : \ (S_j*(-A))(x) \ge 2^{-j}c_jc^{-1}|A|\}\,.$$

By Theorem 3.1 the following holds

$$\dim(B_1) \ll \max\{|S_j|,|A|\}\cdot|A|^{-1}\frac{2^jc}{c_j}\cdot\log|A|\,.$$

Since $(c2^{j-2})^2|B|^2|S_j| \le c_j|A||B|^2$ it follows that $|S_j| \le 16\cdot 2^{-2j}c_jc^{-2}|A|$. If $\max\{|S_j|,|A|\} = |S_j|$; then

$$\dim(B_1) \ll c^{-1}2^{-j}\log|A| \ll c^{-1}\log|A|\,.$$

Now consider the case $\max\{|S_j|,|A|\} = |A|$. We have

$$\dim(B_1) \ll 2^j s\log|A| \ll c^{-1}\log(c^{-1})\log|A|\,.$$

Since

$$\sum_{x\in S_j}(A*B_1)(x) \ge 2^{-j}c_jc^{-1}|A||B|$$

it follows that

$$\sum_{x\in S_j}(A*B_1)(x)(A*B)(x) \ge 2^{-2}c_j|A||B|^2\,.$$

Here the definition of S_j was used. By the Cauchy–Schwartz inequality and the definition of c_j, we obtain

$$E(A,B_1) \ge 2^{-4}s^{-1}E(A,B) \gg \log^{-1}(c^{-1})\cdot E(A,B)\,.$$

This completes the proof.

We now generalize theorem 3.1 to the case of more than two sets.

Theorem 3.6 *Let* $\mathbf{G}$ *be a finite Abelian group,* $k \ge 2$ *be a positive integer,* $A_1,\dots,A_k \subseteq \mathbf{G}$*,* $|A_1| \le |A_2| \le \dots \le |A_k|$ *be sets, and* $\sigma \ge 1$ *be a real number. Let*

$$S = \{x \in \mathbf{G} \ : \ (A_1*\dots*A_{k-2}*A_k*(-A_{k-1}))(x) \ge \sigma\}\,.$$

Then

$$\dim(S) \ll |A_1| \dots |A_{k-2}||A_k| \cdot \sigma^{-1} \cdot \log |A_{k-1}| \,. \qquad (12)$$

Proof. Let $\Lambda \subseteq S$ be the maximal dissociated subset. Consider a simple bipartite graph $G = (V, E)$ with parts $A := A_1 \times \dots \times A_{k-2} \times A_k$ and $B := A_{k-1}$ and colors $\lambda \in \Lambda$ on edges. A vertex $(a_1, \dots, a_{k-2}, a_k) \in A$ is connected to a vertex $b \in B$ by an edge colored $\lambda \in \Lambda$ if and only if $a_1 + \dots a_{k-2} + a_k - b = \lambda$. Note that $|E| \geq \sigma|\Lambda|$. Also note that all edges incident to a certain vertex $a \in A$ have different colors. Finally observe that for any vertex $b \in B$, there exist edges of at least $\deg(b)/(|A_1| \dots |A_{k-2}|)$ different colors that are incident to b. The latter observation follows from the fact that

$$\max_{\lambda \in \Lambda,\, b \in B} (A_1 * \dots * A_{k-2} * A_k)(\lambda + b) \leq |A_1| \dots |A_{k-2}| \,. \qquad (13)$$

To proceed we need in a simple generalization of the Erdös lemma.

Lemma 3.7 *Let $\Gamma = (V, E)$ be a finite simple bipartite graph with parts V_1 and V_2. Suppose d_1, d_2 are positive integers such that $|E| > (d_1 - 1)|V_1| + (d_2 - 1)|V_2|$; then Γ has a bipartite subgraph with parts $V_1' \subseteq V_1$, $V_2' \subseteq V_2$ such that*

$$\min_{v_1' \in V_1'} \deg(v_1') \geq d_1 \,, \quad \text{and} \quad \min_{v_2' \in V_2'} \deg(v_2') \geq d_2 \,.$$

Proof. Take any minimal bipartite subgraph $\Gamma' = (V', E')$ of Γ such that $|E(\Gamma')| > (d_1 - 1)|V_1(\Gamma')| + (d_2 - 1)|V_2(\Gamma')|$, where $V_1(\Gamma') \subseteq V_1$, $V_2(\Gamma') \subseteq V_2$ are the parts of Γ'. It is easy to see that Γ' has the required properties. This completes the proof of the lemma.

To prove theorem 3.6 we apply the generalized Erdös' lemma to G, and obtain a bipartite subgraph G' with parts $A' \subseteq A$, $B' \subseteq B$ such that for all $a' \in A'$ and $b' \in B'$, $\deg(a') \geq 2^{-2}\sigma|\Lambda|/(|A_1| \dots |A_{k-2}||A_k|)$ and $\deg(b') \geq 2^{-2}\sigma|\Lambda|/|A_{k-1}|$. Next we apply the (tree construction) argument from the proof of theorem 3.1 to the graph G'. It is not hard to see that argument yields a non-trivial dependency between the elements of Λ provided

$$\frac{\sigma|\Lambda|}{|A_1| \dots |A_{k-2}||A_k|} \gg \log |A_{k-1}|$$

and

$$\frac{\sigma|\Lambda|}{|A_{k-1}||A_1| \dots |A_{k-2}|} \gg \log |A_{k-1}|$$

This concludes the proof.

We now give our third proof of theorem 1.3. In fact we prove a slightly stronger result (see the inequality (14) below).

Proof of Theorem 1.3. Without a loss of generality assume $|A| \geq |B|$. By assumption $E(A, B) \geq c|A||B|^2$. It follows that

$$\sum_x (B * A * (-A))(x) B_1(x) \geq 2^{-1} c|A||B|^2 \,, \qquad (14)$$

where $B_1 = \{x \in B \ : \ (B * A * (-A))(x) \geq 2^{-1}c|A||B|\}$. Theorem 3.6 yields $\dim(B_1) \ll c^{-1} \log |A|$. Combining the inequality (14) and the Cauchy–Schwartz inequality, we get $E(A, B_1) \gg 2^{-2}c|A||B|^2$ and the theorem follows.

If in theorem 3.6 some extra information on the additive energy of the sets A_j is available; then the bound (12) can be refined for $k \geq 3$ (see [11]). The example of Note 3.5 shows that the corresponding estimates in [11] are sharp.

Note 3.8 Let k be a positive integer and $\Lambda = \{\lambda_1, \dots, \lambda_t\} \subseteq \mathbf{G}$ be a set. We say that Λ belongs to the family $\mathbf{\Lambda}(k)$ if any identity of the form

$$\sum_{j=1}^{t} \varepsilon_j \lambda_j = 0\,, \qquad \varepsilon_j \in \{0, \pm 1\}\,, \qquad \sum_{j=1}^{t} |\varepsilon_j| \leq k\,,$$

yields $\varepsilon_j = 0$, $j \in [t]$. For $E \subseteq \mathbf{G}$ let $\dim_k(E)$ denote the cardinality of the largest subset of E that belongs to the family $\mathbf{\Lambda}(k)$. We remark that the results above will still hold if one replaces $\dim(S)$ with $\dim_k(S)$, say, for $k = O(\log|\mathbf{G}|)$. (For Theorem 1.3 see [8]).

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