Appendix B: DeSitter Cartography and first-Order Gravity

- We go back to the FS approach in Chapter XII.

- deSitter space is there described in terms of a “pure SO(4,1) gauge potential”; the field strength F vanishes.

- This leads to another deSitter cartography using only spinors.

- Contract the SO(4,1) potential A into gamma matrices:

  \[ A^{A \circ B}_\mu \Rightarrow \sum_{A, B} \gamma_A \gamma_B A^{A \circ B}_\mu \equiv A_\mu \]

- Then the “pure gauge” form for A implies that it can be written as follows:

  \[ A_\mu = U \gamma \varphi U \]

- The U’s are an array of four “basis spinors”. I am told that they are known in the trade as “Killing spinors”, or more specifically as “parallel spinors”.
Review of the FS Construction

- The FS field strength $F$ is also contracted into gamma matrices, as a consequence of the construction for $A$.

- The FS action is compactly expressed in this language, especially after the B’s are "integrated out". It is a linear combination of the following three terms:

  Pontryagin + Nieh-Yan:
  \[ L_1 = \text{Tr} \ F \wedge F \]

  Einstein-Hilbert + Euler + cosmological constant:
  \[ L_2 = \text{Tr} \ \gamma_5 \ F \wedge F \]

  Nieh-Yan + Holst:
  \[ L_3 = \text{Tr} \ \gamma_5 \ F \wedge \gamma_5 F \]

- My favorite version of the FS formalism:
  \[ L = \text{Tr} \ G \wedge G \quad \text{with} \quad G = \gamma_5 \theta F \]
• Back to the cartography:

• As an example, we again specialize to the FRW structure for $A$ and $F$:

$$A_\mu = \begin{pmatrix} 0 & O & 0 \\ 15 & 0 & a \\ 01 & 0 & -K \\ 23 & 0 & -C \\ t & x \end{pmatrix}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 15 & \dot{a} - NK & -2ac \\ 01 & -K + Na & 2kc \\ 23 & -c & c^2 - K + a^2 \\ tx & yz & \end{pmatrix}$$

• $F = 0$ implies

$$C = 0 \quad \dot{a} = NK \quad K = Na$$

• This is the deSitter expansion:

$$K = a = e^{nt}$$
• In gamma matrix language,

\[ A_t = \chi_0 \chi_5 N \quad A_x = \chi_1 (\chi_0 - \chi_3) e^{NT} \quad A^? = U^{-1} \partial \mu U \]

• We will be searching for the form of U in what follows.

• Different choices of deSitter cartographies will lead to different choices of the matrix U.

• U is in some sense "the fourth root of the metric tensor".

  The metric tensor g is quadratic in the tetrad e.

  The tetrad e in turn is part of the SO(4,1) potential A, which is quadratic in U.

  Perhaps in some sense we are building the spin two gravity theory out of spin ½ building blocks.
Generating the Catalogue of U’s

- This was done by trial and error, and by generalizing some simple examples.

- Outline of what follows:
  
  Toy examples: SO(3)/SO(2) and SO(2,1)/SO(1,1).
  The SO(4,1)/SO(3,1) FRW deSitter metrics.
  The Static deSitter metric.
  The Painleve-Gullstrand metric.
  The Covariant Conformal metric.
  The Eddington-Finkelstein metric.

For some mathematical insights, see Derek Wise, gr-qc/0611154
A Toy Example

- The groups are SO(3) / SO(2)
- The potential $A$ and field strength $F$ are

\[
A_\mu = \begin{pmatrix} 0 \\ \mp \alpha(t) \\ 0 \\ k(t) \end{pmatrix}
\]

\[
F_{\mu\nu} = \begin{pmatrix} 0 \\ \hat{a} - NK \\ 0 \\ k + Na \end{pmatrix}
\]

Note: the metric here is Euclidean.

- The condition $F = 0$ implies

\[
a(t) = \sin Nt, \quad k(t) = \cos Nt
\]

- Therefore the metric is

\[
ds^2 = N^2 dt^2 + \sin^2 Nt \, dx^2 = d\theta^2 + \sin^2 \theta \, d\phi^2
\]

- This is the surface of a sphere.

- The “time evolution” is big bang/big crunch.
• For this low dimensionality the gamma matrices are 2 x 2 Pauli matrices.

\[ \gamma_0 \rightarrow \sigma_3 \quad \gamma_1 \rightarrow \sigma_1 \quad \gamma_2 \rightarrow \sigma_2 \]

• Construction of U is by common sense:

\[ U = e^{-\frac{i\sigma_2}{2}} \times e^{\frac{i\sigma_1 H t}{2}} \]

• --
SO(2,1) / SO(1,1)

- The construction is similar:
  \[ F = 0 \implies K - Na = 0 \]
  \[ \dot{a} - NK = 0 \]

- Three FRW options:
  \[ \begin{align*}
  k &= 0 & a &= e^{nt} & U &= e^{\frac{x}{2}(\gamma_0 \gamma_1 + \gamma_1 \gamma_2)} e^{\frac{nt}{2} \gamma_2 \gamma_0} \\
  k &= +1 & a &= \cosh nt & U &= e^{\frac{x}{2} \gamma_1 \gamma_2} e^{\frac{nt}{2} \gamma_2 \gamma_0} \\
  k &= -1 & a &= \sinh nt & U &= e^{\frac{x}{2} \gamma_0 \gamma_1} e^{\frac{nt}{2} \gamma_2 \gamma_0}
  \end{align*} \]

- The \( k = 0 \) case is of special interest, because the operator in the exponent is a projection operator:
  \((\gamma_0 \gamma_1 + \gamma_1 \gamma_2)^2 = 0\)

- \((k = 0)\)
  \[ U = \left[ 1 + \frac{x}{2}(\gamma_0 \gamma_1 + \gamma_1 \gamma_2) \right] e^{\frac{nt}{2} \gamma_2 \gamma_0} \]
SO(4,1) / SO(3,1)

- The $k = 0$ generalization is found by analogy:

\[ U = \left[ 1 + \frac{\vec{y} \cdot \vec{x}}{2}(\gamma_5 - \gamma_0) \right] e^{\frac{nt}{2} \gamma_5 \gamma_0} \]

- The $k = 1$ and $k = -1$ cases are obtained by multiplying by "angular factors", which will be ubiquitous:

  \[ k = +1 \quad U = e^{\gamma_2 \gamma_3 \frac{y}{2}} e^{\gamma_2 \gamma_4} e^{\gamma_5 \gamma_0 \frac{x}{2}} e^{\gamma_5 \gamma_0 \frac{nt}{2}} \]

  \[ k = -1 \quad U = e^{\gamma_2 \gamma_3 \frac{y}{2}} e^{\gamma_2 \gamma_4} e^{\gamma_5 \gamma_0 \frac{x}{2}} e^{\gamma_5 \gamma_0 \frac{nt}{2}} \]
• Note that for \( k = \pm 1 \) the lower 3 x 4 part of \( A \) is nonvanishing; this is not torsion, but rather spatial curvature which is contributing.

• The conformally flat cases are obtained by a simple change of time variable:

\[
e^{\frac{\gamma_5 y_0}{2} \frac{nt}{2}} \Rightarrow e^{\frac{\gamma_5 y_0}{2} \ln\left(-\frac{1}{nt}\right)}
\]

• The static deSitter case is obtained from \( k = \pm 1 \) by keeping the angular factors in place but permuting the other two factors:

\[
U = e^{\gamma_2 \gamma_3 \frac{\gamma_2}{2}} e^{\gamma_1 \gamma_2 \frac{y_0}{2}} e^{\gamma_5 y_0 \frac{nt}{2}} e^{\gamma_5 \gamma_3 \frac{x}{2}}
\]

• \[
\begin{bmatrix}
\cos^2 x & dt^2 - dx^2 - \sin^2 x \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)
\end{bmatrix}
\]

\( N = 1 \)
The Painleve-Gullstrand Metric

- Do the SO(2,1) / SO(1,1) case first:

- General form for the gauge potential A:
  \[ A_\mu = \begin{pmatrix} 1 & 0 \\ -x & 1 \\ k(x) & k(x) \\ t & x \end{pmatrix} \]

- Compute F and set it to zero. The result is
  \[ \frac{F}{k} = -x, \quad k = +1 \]

- Then guess the answer for U:
  \[ U = e^{\frac{t}{2} \gamma_0} e^{\frac{t}{2} (\gamma_2 + \gamma_3 - \gamma_0)} = e^{\frac{t}{2} \gamma_0} \left[ 1 + \gamma_1 (\gamma_2 - \gamma_0) \frac{x}{2} \right] \]

- This form of U invites the correct generalization:
  \[ U = e^{\frac{t}{2} \gamma_5 \gamma_0} \left[ 1 + \frac{x}{2} (\gamma_5 - \gamma_0) \right] \]
The Covariant Conformal Metric

• Again, go to SO(2,1)/SO(1,1) first:

• General form for A:

\[ A_\mu = \frac{2\sigma}{12} \begin{pmatrix} a(t) & 0 \\ 0 & a(t) \\ k(t,x) & k(t,x) \end{pmatrix} \]

• The \( F = 0 \) conditions are:

\[ \frac{\tau}{t} a' + ak = 0, \quad \frac{t}{\tau} a' - ak = 0, \quad \frac{2k}{\partial t} - \frac{\partial k}{\partial x} - a^2 = 0 \]

• We know part of the answer: \( a = 1 / (1 - \tau^2) \). Therefore

\[ A_\mu = \frac{2\sigma}{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x & t \end{pmatrix} \cdot \frac{1}{(1-\tau^2)} \]

• From this, guess \( U \):

\[ U = \frac{1 + \gamma_2 (\gamma_2 t - \gamma_1 x)}{\sqrt{1-\tau^2}} \]
The generalization is direct and pretty:

$$\mathcal{U} = \frac{1}{\sqrt{1-t^2}} \left[ 1 + \gamma_5 (\gamma_0 t - \gamma_0 \gamma) \right]$$

The explicit expression for $A$ is also pretty:

$$A_p = \begin{pmatrix}
50 & 1 & 0 & 0 & 0 \\
15 & 0 & 1 & 0 & 0 \\
25 & 0 & 0 & 1 & 0 \\
35 & 0 & 0 & 0 & 1 \\
01 & \times & t & 0 & 0 \\
02 & \times & 0 & t & 0 \\
03 & \times & 0 & 0 & t \\
23 & o & 0 & -z & y \\
31 & o & z & 0 & -x \\
12 & o & y & x & 0 \\
t & x & y & z
\end{pmatrix} \cdot \frac{1}{1-t^2}$$
The Eddington-Finkelstein Metric

• Again, retreat to $SO(2,1) / SO(1,1)$ first:

• The metric is

\[ ds^2 = dt^2 - dx^2 - x^2(\frac{dx}{dt} - dx)^2 \]

• The “dyad” is

\[ C_\mu = \begin{pmatrix}
1 & x \\
\frac{x}{1-x} & 1 + \frac{x^2}{1-x}
\end{pmatrix} \]

• The answer is not worked out.
## Summary

<table>
<thead>
<tr>
<th></th>
<th>((u, v, w))</th>
<th>Metric</th>
<th>Parallel spinors</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Covariant</strong></td>
<td>(t = \frac{v}{w+1})</td>
<td>(ds^2 = \frac{1}{(1-v^2)^2} (dt^2 - dx^2 - dy^2 - dz^2))</td>
<td>(U = \left[1 + \gamma_5 (\gamma_0 t - \gamma_0 x)\right])</td>
</tr>
<tr>
<td><strong>Conformal</strong></td>
<td>(x = \frac{r^2}{w+1})</td>
<td>(\gamma^2 = t^2 - x^2 - y^2 - z^2)</td>
<td></td>
</tr>
<tr>
<td><strong>Painleve-Gullstrand</strong></td>
<td>(w + v = e^t)</td>
<td>(ds^2 = dt^2 - (dx - x dt)^2 - (dy - y dt)^2 (dz^2))</td>
<td>(U = e^{-\frac{r^0}{2}} \left[1 + \frac{r^0}{2} (\gamma_5 - \gamma_0)\right])</td>
</tr>
<tr>
<td><strong>Eddington-Finkelstein</strong></td>
<td>(w + v = (1 + r)e^{(t-r)})</td>
<td>(ds^2 = dt^2 - dr^2 - r^2 [dt^2 - dr^2 + dy + \sin y dz^2])</td>
<td>(U = ?)</td>
</tr>
<tr>
<td><strong>Static</strong></td>
<td>(r = \sin x)</td>
<td>(ds^2 = \cos^2 x dt^2 - dx^2 - \sin^2 x (dy + \sin y dz^2))</td>
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<tr>
<td></td>
<td>(w = \cos x \cosh t)</td>
<td>(= (1-r^2) dt^2 - \frac{dr^2}{(1-r^2)} - r^2 (dy + \sin y dz^2))</td>
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<tr>
<td></td>
<td>(v = \cos x \sinh t)</td>
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<td></td>
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<tr>
<td></td>
<td></td>
<td>(U = e^{\frac{r^0 r^2}{2}} \gamma_0 \gamma_2 \gamma_4 \gamma_{\frac{1}{2}} \gamma_{\frac{3}{2}})</td>
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</table>

The table above summarizes various metrics and parallel spinors in different coordinate systems.
# Summary (continued)

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<tr>
<td>Closed FRW ((k = +1))</td>
<td>(v = \sinh t) &lt;br&gt; (w = \cosh t \cos x) &lt;br&gt; (r = \cosh t \sin x)</td>
<td>[ds^2 = dt^2 - \cosh t \left[ dx^2 + \sinh x (dy^2 + \sin y , dz^2) \right] ] &lt;br&gt; [= dt^2 - \cosh t \left[ \frac{dr^2}{(1-r^2)} + r^2 (dy^2 + \sin^2 y , dz^2) \right] ]</td>
<td>(U = e , e , \bar{e} , \bar{e} ) &lt;br&gt; (\gamma_{3/2} , \gamma_{4/2} , \gamma_{5/2} , \gamma_{5/2} ) &lt;br&gt; (\gamma_{5/2} , \gamma_{5/2} )</td>
</tr>
<tr>
<td>Open FRW ((k = -1))</td>
<td>(W = \cosh t) &lt;br&gt; (V = \sinh t \cosh x) &lt;br&gt; (r = \sinh t \sin x)</td>
<td>[ds^2 = dt^2 - \sinh t \left[ dx^2 + \sinh x (dy^2 + \sin y , dz^2) \right] ] &lt;br&gt; [= dt^2 - \sinh t \left[ \frac{dr^2}{(1+r^2)} + r^2 (dy^2 + \sin^2 y , dz^2) \right] ]</td>
<td>(U = e , \bar{e} , e , \bar{e} ) &lt;br&gt; (\gamma_{3/2} , \gamma_{4/2} , \gamma_{5/2} , \gamma_{5/2} ) &lt;br&gt; (\gamma_{5/2} , \gamma_{5/2} )</td>
</tr>
<tr>
<td>Flat FRW ((k = 0))</td>
<td>(w+v = e^t) &lt;br&gt; (w-v = e^{-t} - p_e^t) &lt;br&gt; (r = p_e t)</td>
<td>[ds^2 = dt^2 - e^{2t} (dx^2 + dy^2 + dz^2)] &lt;br&gt; (p^2 = (x^2 + y^2 + z^2))</td>
<td>(U = \left[ 1 + \frac{\gamma_{x} \times (\gamma_{5} - \gamma_{0})}{2} \right] \gamma_{5/2} \gamma_{5/2} )</td>
</tr>
<tr>
<td>Flat Conformal</td>
<td>(w+v = -\frac{1}{t}) &lt;br&gt; (w-v = -t + \frac{p_e^2}{t}) &lt;br&gt; (r = -\frac{p_e}{t})</td>
<td>[ds^2 = \frac{1}{t^2} (dt^2 - dx^2 - dy^2 - dz^2)] &lt;br&gt; (p^2 = (x^2 + y^2 + z^2))</td>
<td>(U = \left[ 1 + \frac{\gamma_{x} \times (\gamma_{5} - \gamma_{0})}{2} \right] (-\frac{1}{t}) \gamma_{5/2} \gamma_{5/2} )</td>
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