Appendix B: DeSitter Cartography and first-Order Gravity

- We go back to the FS approach in Chapter XII.
- deSitter space is there described in terms of a "pure SO(4,1) gauge potential"; the field strength F vanishes.
- This leads to another deSitter cartography using only spinors.
- Contract the SO(4,1) potential A into gamma matrices:

$$A_\mu^{AB} \Longrightarrow \sum_{A,B} \gamma_A \gamma_B A_\mu^{AB} \equiv A_\mu$$

- Then the "pure gauge" form for A implies that it can be written as follows:

$$A_\mu = U^{-1} \partial_\mu U$$

- The U's are an array of four "basis spinors". I am told that they are known in the trade as "Killing spinors", or more specifically as "parallel spinors".

Review of the FS Construction

- The FS field strength F is also contracted into gamma matrices, as a consequence of the construction for A.

- The FS action is compactly expressed in this language, especially after the B's are "integrated out" . It is a linear combination of the following three terms:

  Pontryagin + Nieh-Yan:

$$\mathcal{L}_1 = \mathrm{Tr}\ F \wedge F$$

  Einstein-Hilbert + Euler + cosmological constant:

$$\mathcal{L}_2 = \mathrm{Tr}\ \gamma_5 F \wedge F$$

  Nieh-Yan + Holst:

$$\mathcal{L}_3 = \mathrm{Tr}\ \gamma_5 F \wedge \gamma_5 F$$

- My favorite version of the FS formalism:

$$\mathcal{L} = \mathrm{Tr}\ G \wedge G \quad \text{with} \quad G = e^{\gamma_5 \theta} F$$

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- Back to the cartography:

- As an example, we again specialize to the FRW structure for A and F:

$$A_\mu = \begin{array}{c} 05 \\ 15 \\ 01 \\ 23 \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} N & 0 \\ 0 & a \\ 0 & -K \\ 0 & -C \end{pmatrix} \\ \begin{array}{cc} t & \;\; x \end{array} \end{array} \qquad F_{\mu\nu} = \begin{array}{c} 05 \\ 15 \\ 01 \\ 23 \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} 0 & 0 \\ \mathring{a} - NK & -2aC \\ -\mathring{K} + Na & 2KC \\ -\mathring{C} & C^2 - K^2 + a^2 \end{pmatrix} \\ \begin{array}{cc} tx & \qquad yz \end{array} \end{array}$$

- F = 0 implies

$$C = 0 \qquad \mathring{a} = NK \qquad \mathring{K} = Na$$

- This is the deSitter expansion: $K = a = e^{Nt}$

- In gamma matrix language,

$$A_t = \gamma_0 \gamma_5 N \qquad A_x = \gamma_1 (\gamma_0 - \gamma_5) e^{Nt} \qquad A_\mu \overset{?}{=} U^{-1} \partial_\mu U$$

- We will be searching for the form of U in what follows.

- Different choices of deSitter cartographies will lead to different choices of the matrix U.

- U is in some sense “the fourth root of the metric tensor”.

The metric tensor g is quadratic in the tetrad e.

The tetrad e in turn is part of the SO(4,1) potential A, which is quadratic in U.

Perhaps in some sense we are building the spin two gravity theory out of spin ½ building blocks.

Generating the Catalogue of U's

- This was done by trial and error, and by generalizing some simple examples.

- Outline of what follows:

  Toy examples: SO(3)/SO(2) and SO(2,1)/SO(1,1).
  The SO(4,1)/SO(3,1) FRW deSitter metrics.
  The Static deSitter metric.
  The Painleve-Gullstrand metric.
  The Covariant Conformal metric.
  The Eddington-Finkelstein metric.

For some mathematical insights, see Derek Wise, gr-qc/0611154

A Toy Example

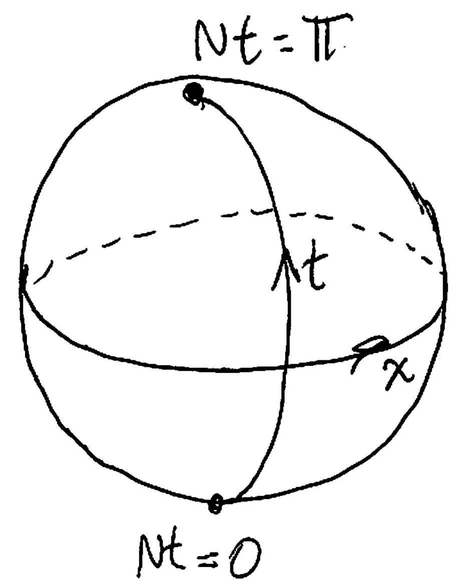
- The groups are SO(3) / SO(2)
- The potential A and field strength F are

$$A_\mu = \begin{array}{c c} \begin{array}{c} 20 \\ 12 \\ 01 \end{array} & \left(\begin{array}{c c} N & 0 \\ 0 & a(t) \\ 0 & K(t) \end{array}\right) \\ & \begin{array}{c c} \;t\; & \;\;x\;\; \end{array} \end{array} \qquad F_{tx} = \begin{array}{c} 20 \\ 12 \\ 01 \end{array} \left(\begin{array}{c} 0 \\ \dot{a} - NK \\ \dot{K} + Na \end{array}\right)$$

Note: the metric here is Euclidean.

- The condition F = 0 implies

$$a(t) = \sin Nt \qquad K(t) = \cos Nt$$



- Therefore the metric is

$$ds^2 = N^2 dt^2 + \sin^2 Nt \, dx^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

- This is the surface of a sphere.
- The "time evolution" is big bang/big crunch.

- For this low dimensionality the gamma matrices are 2 x 2 Pauli matrices.

$$\gamma_0 \to \sigma_3 \qquad \gamma_1 \to \sigma_1 \qquad \gamma_2 \to \sigma_2$$

- Construction of U is by common sense:

$$U = e^{-\frac{i\sigma_2}{2}x}\, e^{-\frac{i\sigma_1 Nt}{2}}$$

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SO(2,1) / SO(1,1)

- The construction is similar:

$$F = 0 \quad \Rightarrow \quad \begin{aligned} \mathring{K} - Na &= 0 \\ \mathring{a} - NK &= 0 \end{aligned}$$

- Three FRW options:

$$k = 0 \qquad a = e^{Nt} \qquad U = e^{\frac{x}{2}(\gamma_0\gamma_1 + \gamma_1\gamma_2)}\, e^{\frac{Nt}{2}\gamma_2\gamma_0}$$

$$k = +1 \qquad a = \cosh Nt \qquad U = e^{\frac{x}{2}\gamma_1\gamma_2}\, e^{\frac{Nt}{2}\gamma_2\gamma_0}$$

$$k = -1 \qquad a = \sinh Nt \qquad U = e^{\frac{x}{2}\gamma_0\gamma_1}\, e^{\frac{Nt}{2}\gamma_2\gamma_0}$$

- The k = 0 case is of special interest, because the operator in the exponent is a projection operator: $(\gamma_0\gamma_1 + \gamma_1\gamma_2)^2 = 0$

- -- $(k = 0)$ $\quad U = \left[1 + \frac{x}{2}(\gamma_0\gamma_1 + \gamma_1\gamma_2)\right] e^{\frac{Nt}{2}\gamma_2\gamma_0}$

SO(4,1) / SO(3,1)

- The k = 0 generalization is found by analogy:

$$U = \left[1 + \frac{\vec{\gamma}\cdot\vec{x}}{2}(\gamma_5 - \gamma_0)\right] e^{\frac{Nt}{2}\gamma_5\gamma_0}$$

- The k = 1 and k = -1 cases are obtained by multiplying by "angular factors", which will be ubiquitous:

$$k = +1 \qquad U = e^{\gamma_2\gamma_3\frac{z}{2}}\, e^{\gamma_1\gamma_2\frac{y}{2}}\, e^{\gamma_1\gamma_5\frac{x}{2}}\, e^{\gamma_5\gamma_0\frac{Nt}{2}}$$

$$k = -1 \qquad U = e^{\gamma_2\gamma_3\frac{z}{2}}\, e^{\gamma_1\gamma_2\frac{y}{2}}\, e^{\gamma_0\gamma_1\frac{x}{2}}\, e^{\gamma_5\gamma_0\frac{Nt}{2}}$$

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- Note that for k = ± 1 the lower 3 x 4 part of A is nonvanishing; this is not torsion, but rather spatial curvature which is contributing.

- The conformally flat cases are obtained by a simple change of time variable:

$$e^{\gamma_5\gamma_0 \frac{Nt}{2}} \Rightarrow e^{\frac{\gamma_5\gamma_0}{2} \ln\left(-\frac{1}{Nt}\right)}$$

- The static deSitter case is obtained from k = ± 1 by keeping the angular factors in place but permuting the other two factors:

$$U = e^{\gamma_2\gamma_3 \frac{z}{2}}\, e^{\gamma_1\gamma_2 \frac{y}{2}}\, e^{\gamma_5\gamma_0 \frac{Nt}{2}}\, e^{\gamma_1\gamma_5 \frac{x}{2}}$$

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$$\left[\, ds^2 = \cos^2\chi \underset{\underset{Nt}{\uparrow}}{dt^2} - \underset{\underset{x}{\uparrow}}{d\chi^2} - \sin^2\chi\,(\underset{\underset{y}{\uparrow}}{d\theta^2} + \sin^2\theta\, \underset{\underset{z}{\uparrow}}{d\varphi^2})\,\right] \quad N=1$$

The Painleve-Gullstrand Metric

- Do the SO(2,1) / SO(1,1) case first:
- General form for the gauge potential A:

$$
A_\mu = \begin{array}{c c} & \begin{array}{cc} \quad t & \quad x \end{array} \\ \begin{array}{c} 02 \\ 12 \\ 01 \end{array} & \begin{pmatrix} 1 & 0 \\ -x & 1 \\ k(x) & K(x) \end{pmatrix} \end{array}
$$

- Compute F and set it to zero. The result is

$$
k = -x
$$
$$
K = +1
$$

- Then guess the answer for U:

$$
U = e^{\frac{t}{2}\gamma_0\gamma_2}\, e^{\frac{x}{2}(\gamma_1\gamma_2 + \gamma_0\gamma_1)} = e^{\frac{t}{2}\gamma_0\gamma_2}\left[1 + \gamma_1(\gamma_2 - \gamma_0)\frac{x}{2}\right]
$$

- This form of U invites the correct generalization:

$$
U = e^{\frac{t}{2}\gamma_5\gamma_0}\left[1 + \frac{\vec{\gamma}\cdot\vec{x}}{2}(\gamma_5 - \gamma_0)\right]
$$

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The Covariant Conformal Metric

- Again, go to SO(2,1) /SO(1,1) first:

- General form for A:

$$A_\mu = \begin{array}{c c} \begin{matrix} 20 \\ 12 \\ 01 \end{matrix} & \begin{pmatrix} a(\tau) & 0 \\ 0 & a(\tau) \\ k(t,x) & K(t,x) \end{pmatrix} \\ & \begin{matrix} \quad t & \qquad\quad x \end{matrix} \end{array}$$

- The F = 0 conditions are;

$$\frac{x}{\tau} a' + a k = 0 \qquad \frac{t}{\tau} a' - a K = 0 \qquad \frac{\partial K}{\partial t} - \frac{\partial k}{\partial x} - a^2 = 0$$

- We know part of the answer: a = 1 / (1 – τ^2). Therefore

$$A_\mu = \begin{array}{c c} \begin{matrix} 20 \\ 12 \\ 01 \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x & t \end{pmatrix} \end{array} \cdot \frac{1}{(1-\tau^2)}$$

- From this, guess U:

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$$U = \frac{1 + \gamma_2(\gamma_0 t - \gamma_1 x)}{\sqrt{1-\tau^2}}$$

- The generalization is direct and pretty:

$$U = \frac{1}{\sqrt{1-\tau^2}}\left[1 + \gamma_5\left(\gamma_0 t - \vec{\gamma}\cdot\vec{x}\right)\right]$$

- The explicit expression for A is also pretty:

$$A_\mu = \begin{array}{cc} \begin{matrix} 50 \\ 15 \\ 25 \\ 35 \\ 01 \\ 02 \\ 03 \\ 23 \\ 31 \\ 12 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -x & t & 0 & 0 \\ -y & 0 & t & 0 \\ -z & 0 & 0 & t \\ 0 & 0 & -z & y \\ 0 & z & 0 & -x \\ 0 & -y & x & 0 \end{pmatrix} \\ & \begin{matrix} t & x & y & z \end{matrix} \end{array} \cdot \frac{1}{(1-\tau^2)}$$

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The Eddington-Finkelstein Metric

- Again, retreat to SO(2,1) / SO(1,1) first:

- The metric is $ds^2 = dt^2 - dx^2 - x^2(dt - dx)^2$

- The "dyad" is

$$e_\mu = \begin{array}{cc} 20 \\ 21 \\ \end{array}\!\!\begin{array}{c} \begin{pmatrix} 1 & \frac{x}{1-x} \\ x & 1 + \frac{x^2}{1-x} \end{pmatrix} \\ \begin{array}{cc} t & \qquad x \end{array} \end{array}$$

- The answer is not worked out.

Summary

	(u, v, w)	Metric	Parallel spinors
Covariant Conformal	$t = \frac{v}{w+1}$ $\vec{x} = \frac{\vec{r}}{w+1}$	$ds^2 = \frac{1}{(1-\tau^2)^2}\left(dt^2 - dx^2 - dy^2 - dz^2\right)$ $\tau^2 = t^2 - x^2 - y^2 - z^2$	$U = \frac{\left[1 + \gamma_5(\gamma_0 t - \vec{\gamma}\cdot\vec{x})\right]}{\sqrt{(1-\tau^2)}}$
Painleve-Gullstrand	$w + v = e^t$ $w \cdot v = e^{-t}(1-r^2)$	$ds^2 = dt^2 - (dx - x\,dt)^2 - (dy - y\,dt)^2 - (dz - z\,dt)^2$	$U = e^{\gamma_5\gamma_0 \frac{t}{2}}\left[1 + \frac{\vec{\gamma}\cdot\vec{x}}{2}(\gamma_5 - \gamma_0)\right]$
Eddington-Finkelstein	$w + v = (1+r)e^{(t-r)}$ $w - v = (1-r)e^{-(t-r)}$	$ds^2 = dt^2 - dr^2 - r^2\left[(dt-dr)^2 + dy^2 + \sin^2 y\, dz^2\right]$	$U = \ ?$
Static	$r = \sin x$ $w = \cos x \cosh t$ $v = \cos x \sinh t$	$ds^2 = \cos^2 x\, dt^2 - dx^2 - \sin^2 x\,(dy^2 + \sin^2 y\, dz^2)$ $= (1-r^2)dt^2 - \frac{dr^2}{(1-r^2)} - r^2(dy^2 + \sin^2 y\, dz^2)$	$U = e^{\gamma_2\gamma_3\frac{z}{2}}\, e^{\gamma_1\gamma_2\frac{y}{2}}\, e^{\gamma_5\gamma_0\frac{t}{2}}\, e^{\gamma_1\gamma_5\frac{x}{2}}$

Summary (continued)

	(u,v,w)	Metric	Parallel Spinors
Closed FRW $(k=+1)$	$v = \sinh t$ $w = \cosh t \cos x$ $r = \cosh t \sin x$	$ds^2 = dt^2 - \cosh^2 t\left[dx^2 + \sin^2 x\left(dy^2 + \sin^2 y\, dz^2\right)\right]$ $= dt^2 - \cosh^2 t\left[\frac{dr^2}{(1-r^2)} + r^2\left(dy^2 + \sin^2 y\, dz^2\right)\right]$	$U = e^{\gamma_2\gamma_3\frac{z}{2}}\, e^{\gamma_1\gamma_2\frac{y}{2}}\, e^{\gamma_1\gamma_5\frac{x}{2}}\, e^{\gamma_5\gamma_0\frac{t}{2}}$
Open FRW $(k=-1)$	$w = \cosh t$ $v = \sinh t \cosh x$ $r = \sinh t \sinh x$	$ds^2 = dt^2 - \sinh^2 t\left[dx^2 + \sinh^2 x\left(dy^2 + \sin^2 y\, dz^2\right)\right]$ $= dt^2 - \sinh^2 t\left[\frac{dr^2}{(1+r^2)} + r^2\left(dy^2 + \sin^2 y\, dz^2\right)\right]$	$U = e^{\gamma_2\gamma_3\frac{z}{2}}\, e^{\gamma_1\gamma_2\frac{y}{2}}\, e^{\gamma_0\gamma_1\frac{x}{2}}\, e^{\gamma_5\gamma_0\frac{t}{2}}$
Flat FRW $(k=0)$	$w+v = e^t$ $w-v = e^{-t} - \rho^2 e^t$ $r = \rho e^t$	$ds^2 = dt^2 - e^{2t}\left(dx^2 + dy^2 + dz^2\right)$ $\rho^2 = (x^2 + y^2 + z^2)$	$U = \left[1 + \frac{\vec{\gamma}\cdot\vec{x}}{2}(\gamma_5 - \gamma_0)\right] e^{\gamma_5\gamma_0\frac{t}{2}}$
Flat • -- Conformal	$w+v = -\frac{1}{t}$ $w-v = -t + \frac{\rho^2}{t}$ $r = -\frac{\rho}{t}$	$ds^2 = \frac{1}{t^2}\left(dt^2 - dx^2 - dy^2 - dz^2\right)$ $\rho^2 = (x^2 + y^2 + z^2)$	$U = \left[1 + \frac{\vec{\gamma}\cdot\vec{x}}{2}(\gamma_5 - \gamma_0)\right]\left(-\frac{1}{t}\right)^{\frac{\gamma_5\gamma_0}{2}}$