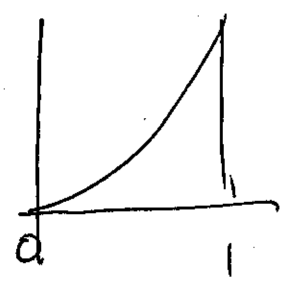
Integrals – Applications

we originally used rectangles to find areas
For example the area under $f(x) = x^2$ on $[0,1]$



so

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n}$$

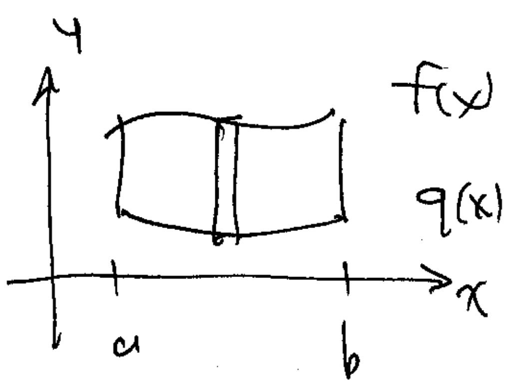
and we defined the definite integral

$$= \int_0^1 x^2 \, dx$$

and via the fundamental Th^m

$$= \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

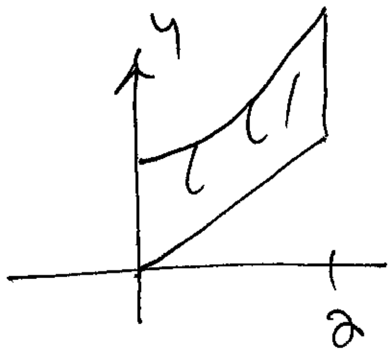
Area Between Curves



To find this ~~so~~ area
we find the area under $f(x)$
find the area under $g(x)$
and subtract so

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \big(f(x) - g(x)\big)\,dx$$

Ex 1 Find the area between $y = 1+x^2$ & $y = x$ on $[0, 2]$



$$A = \int_0^2 (1+x^2-x)\,dx$$

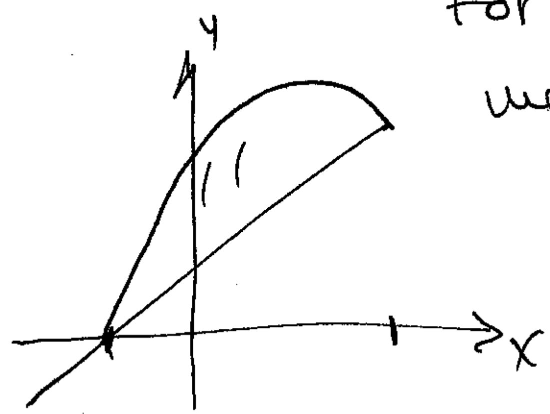
$$= x + \frac{x^3}{3} - \frac{x^2}{2}\Big|_0^2$$

$$= 2 + \frac{8}{3} - 2 = 8/3$$

Ex 2 Find the area between $y = x+1$ &

$y = 3+2x-x^2$. First we sketch the region

For limits of integration we find intersection pt



$$x+1 = 3+2x-x^2$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0 \qquad x = -1, 2$$

$$\int_{-1}^{2} \underbrace{(3+2x-x^2)}_{\text{top curve}} - \underbrace{(x+1)}_{\text{bottom curved}}\,dx = \int_{-1}^{2} (2+x-x^2)\,dx$$

$$= 2x + \frac{x^2}{2} - \frac{x^3}{3}\Big|_{-1}^{2}$$

$$= \left(4+2-\frac{8}{3}\right) - \left(-2+\frac{1}{2}+\frac{1}{3}\right) = 9/2$$

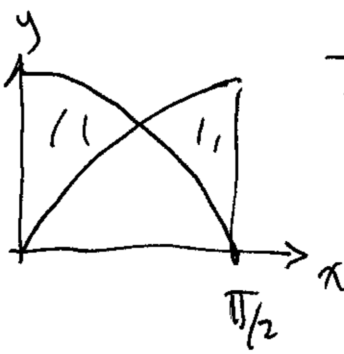
Ex 3 Find the area between $y=\cos x$ & $y=\sin x$ on $[0,\pi/2]$. So

$$A = \int_0^{\pi/2} (\cos x - \sin x)\,dx = \sin x + \cos x \Big|_0^{\pi/2}$$

$$= (\sin \pi/2 + \cos \pi/2) - (\sin 0 + \cos 0) = 1 - 1 = 0$$

0 Area ?

Now we sketch the region



the top curve changes at $x = \pi/4$ so we need 2 $\int$'s

$$A = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4} - (\cos x + \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= (\sin \tfrac{\pi}{4} + \cos \tfrac{\pi}{4} + \sin 0 - \cos 0]$$

$$- (\cos \pi/2 + \sin \pi/2 - \cos \pi/4 - \sin \pi/4)$$

$$= \left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} - 1\right) - \left(1 - \tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}\right)$$

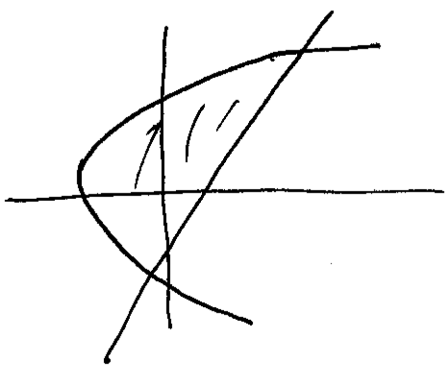
$$= 2\sqrt{2} - 2$$

Ex 4 Find the area between

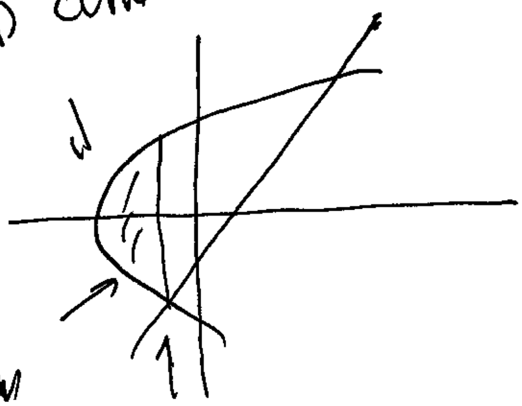
$y = x-1$ & $y^2 = 2x+6$

↑ we would need to solve this for y

$y = \pm\sqrt{2x+6}$



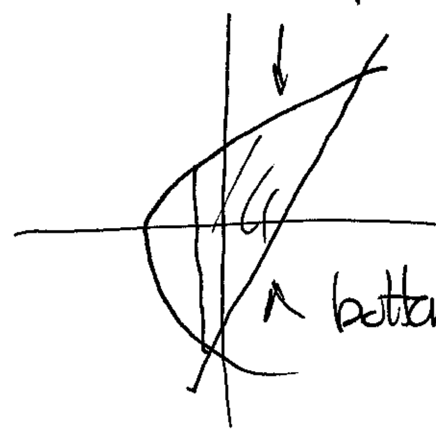
top curve



bottom curve

$x = -1$

top curve



bottom curve

So in this example the bottom curve changes when the curves intersect at $x = -1$

$y^2 = 2(y+1)+6$

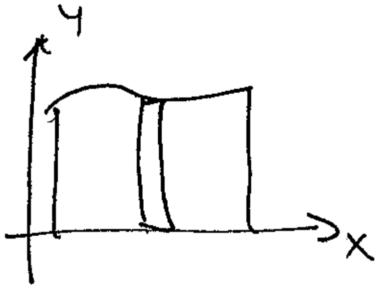
$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

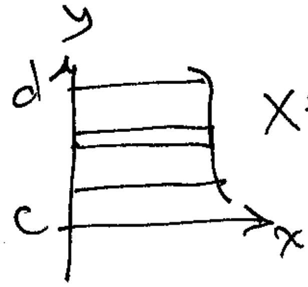
$y = -2, 4$ or $x = -1, 5$

$$A = \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx + \int_{-1}^{5} \sqrt{2x+6} - (x-1)\, dx$$

doable but there's an easier way!

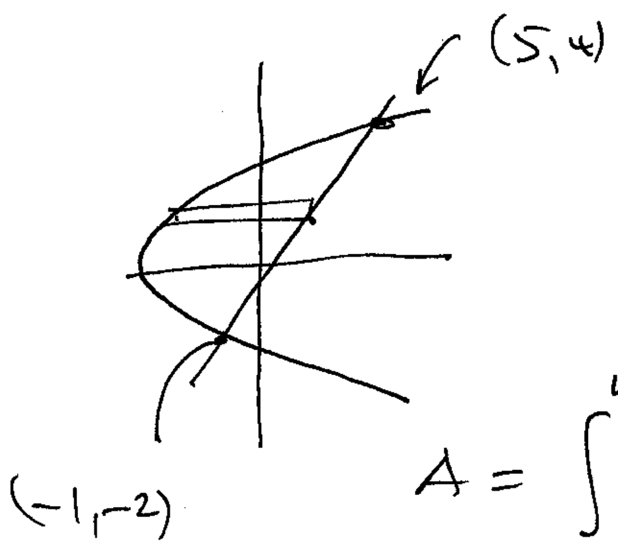


$A = \int_a^b f(x)\,dx$ – vertical rectangles



$x = g(y)$ $A = \int_c^d g(y)\,dy$ – horizontal rectangles

Previous Problem



right curve $y = x - 1$ or $x = y + 1$

left curve $y^2 = 2x + 6$

or $x = \frac{y^2 - 6}{2}$

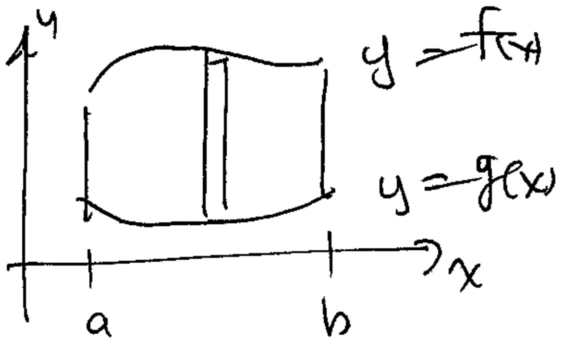
$$A = \int_{-2}^{4} (y+1) - \left(\frac{y^2-6}{2}\right) dy$$

$$= \int_{-2}^{4} 4 + y - \frac{y^2}{2}\,dy = 4y + \frac{y^2}{2} - \frac{y^3}{6}\Big|_{-2}^{4}$$

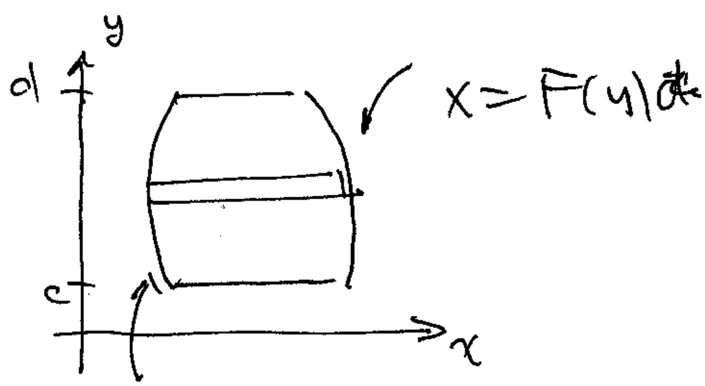
$$= \left(16 + 8 - \frac{32}{3}\right) - \left(-8 + 2 + \frac{4}{3}\right)$$

$= 18$ only 1 integral and easier to $\int$

In general



$$A = \int_a^b f(x) - g(x)\,dx$$



$x = G(y)$

$$A = \int_c^d F(y) - G(y)\,dy$$