Functions

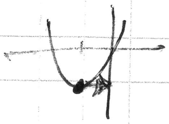
Express $x^2 + 2x$ in the form $(x + h)^2 + k$, where $h, k \in \mathbb{Z}$.
Hence, write down the coordinates of the turning point of the graph of $f(x) = x^2 + 2x$.
Now draw a sketch of the graph of $f(x)$.

$x^2 + 2x$

$(x+1)^2$

$\underbrace{x^2+2x+1}$ $-1 = x^2+2x$

$(x+1)^2 - 1 = x^2+2x$



$(x+1)^2 - 1$

$((-1)+1)^2 - 1$

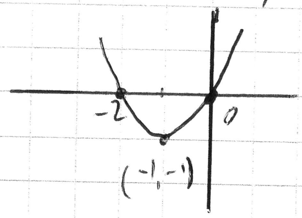
-1

$(-1, -1)$

Sketch $x^2+2x = 0$

$x(x+2) = 0$

$x = 0, \ x = -2$



or $x^2 + 2x = (x+h)^2 + k$

$\boxed{x^2} + 2x = \boxed{x^2} + 2xh + h^2 + k$

$2x = 2xh$

$2 = 2h$

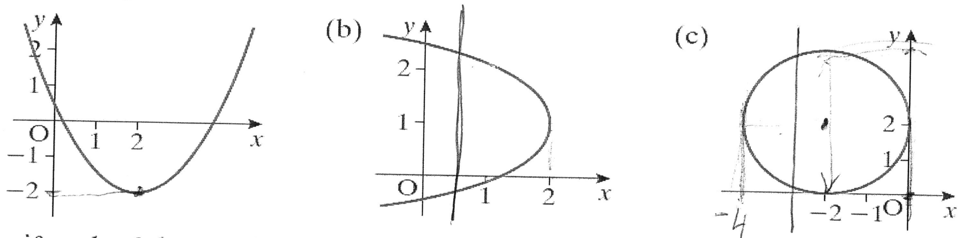
$1 = h$

$0 = h^2 + k$

$0 = (1)^2 + k$

$-1 = k$

(i) State the domain and range for the relations represented by each of the following graphs:



(ii) State if each of the graphs represents a function, giving a reason for your answer in each case.

(i) a) Real Numbers ⇒ Domain b) $x \le 2$ ⇒ Domain c) $-4 \le x \le 0$ ⇒ Domain
$y \geq -2$ ⇒ Range   Real No.s ⇒ Range $0 \le x \le 4$ ⇒ Range.

ii) a) is a function, in a vertical line test, a line would cut only once.
b) is not a function since in a vertical line test, a line would cut more than once.
c) same as b)

Given $f(x) = 10x$ and $g(x) = x + 3$,

(i) find $fg(x)$ and $(fg)^{-1}(x)$

(ii) prove that if $fg(a) = b$, then $(fg)^{-1}(b) = a$. ✓

(i) $\boxed{fg(x) = 10(x+3)}$

$y = 10x + 30$

$y - 30 = 10x$

$\frac{y-30}{10} = x$

$\frac{x-30}{10} = y$

$\frac{x-30}{10} = fg^{-1}(x)$

(ii) $fg(a) = 10(a+3)$

$fg(a) = 10a + 30 = b$

So $fg^{-1}(b) = \frac{b-30}{10}$

$fg^{-1}(b) = \frac{(10a+30)-30}{10}$

$fg^{-1}(b) = \frac{10a+30-30}{10} = \frac{10a}{10} = a$

The graph of $f(x) = x^3 - 2x^2 - 5x + 6$ is shown.

Use the graph to write down

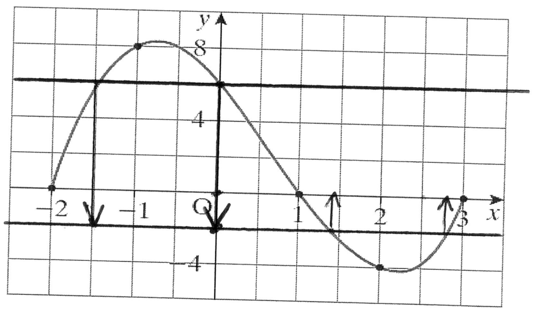
(i) the roots of the equation $f(x) = 0$

(ii) the roots of the equation $f(x) = -2$

(iii) the roots of the equation $x^3 - 2x^2 - 5x = 0$.

Is $y = f(x)$ a one-to-one function? Explain.

Explain why $f(x)$ is a surjective function in the given range.



(i) $f(x) = 0$

$y = 0$ x-axis

$-2, 1, 3$

(ii) $f(x) = -2$

$y = -2$

$1{\cdot}4, 2{\cdot}8$

(iii) $x^3 - 2x^2 - 5x = 0$

$+6 \quad +6$

$\underbrace{x^3 - 2x^2 - 5x + 6} = 6$

$f(x) = 6$

$y = 6$

$-1.5, 0$.

⇒ not one to one mapping as in a horizontal line test, line would cut graph at least once. Hence why f(x) is a surjective function.

The functions f and g are defined as follows:

$$f(x) = x^2,\ x \in \mathbb{R} \text{ and } g(x) = \frac{1}{2x-3},\ \text{for } x \in \mathbb{R}, x \neq \tfrac{3}{2}.$$

(i) State the range of f.

(ii) The inverse of g is g^{-1}. Find $g^{-1}(x)$.

(iii) State the range of g^{-1}.

(iv) Solve the equation $fg(x) = 9$.

(i) Range of f, $y \geq 0$.

(ii) $y = \frac{1}{2x-3}$

$y(2x-3) = \left(\frac{1}{\cancel{2x-3}}\right)\cancel{(2x-3)}$

$2xy - 3y = 1$

$2xy = 1 + 3y$

$x = \frac{1+3y}{2y}$

$y = \frac{1+3x}{2x}$

$g^{-1}(x) = \frac{1+3x}{2x}$

(iii) Range of $g^{-1}(x)$ is same as domain of $g(x)$, so $y \in R$, $y \neq 3/2$

(iv) recap $f(x) = x^2$

$f(1) = (1)^2 = 1$

$f(-3) = (-3)^2 = 9$

$f\boxed{g(x)} = \left(\frac{1}{2x-3}\right)^2$

$9 = \frac{1}{(2x-3)(2x-3)}$

$9 = \frac{1}{4x^2-12x+9}$

$(4x^2-12x+9)9 = \frac{1}{\cancel{(4x^2-12x+9)}}\cancel{(4x^2-12x+9)}$

$36x^2 - 108x + 81 = 1$

$36x^2 - 108x + 80 = 0$

$9x^2 - 27x + 20 = 0$

$(3x-5)(3x-4) = 0$

$3x = 5$ $\quad$ $3x = 4$

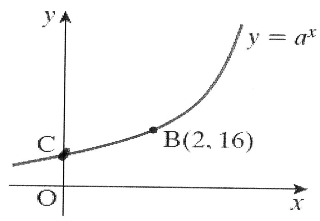
$x = \frac{5}{3}$ or $x = \frac{4}{3}$.

Part of the graph of $y = a^x$, where $a > 0$, is shown. The graph cuts the y-axis at C.

(i) Write down the coordinates of C.

B is the point (2, 16).

(ii) Calculate the value of a.



Exponential function, $3, 9, 27, 81$

$\times 3 \quad \times 3 \quad \times 3$

$T_n = 3^n$

(i) $x = 0$

$y = a^x$

$y = a^0$

$y = 1$

$(0, 1)$

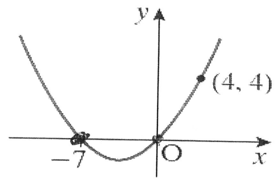
(ii) $y = a^x$ $\quad (2, 16)$ $\quad x, y$

$16 = a^2$

$\sqrt{16} = a$

$4 = a$

Determine the equation of the parabola shown on the right.



$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$x^2 - ((-7)+(0))\,x + ((-7)(0)) = 0$$

$$x^2 - (-7)x + 0 = 0$$

$$k(x^2 + 7x) = 0$$

(4,4)
x, y

$$f(x) = k(x^2 + 7x)$$

$$y = k(x^2 + 7x)$$

$$4 = k((4)^2 + 7(4))$$

$$4 = k(44)$$

$$4 = 44k$$

$$\frac{1}{11} = \frac{4}{44} = k$$

$$f(x) = \frac{1}{11}(x^2 + 7x).$$

Another Way

$$y = ax^2 + bx + c$$

(-7,0)
x y

$$0 = a(-7)^2 + b(-7) + c$$

$$0 = 49a - 7b + c$$

(0,0)
x, y,

$$0 = a(0)^2 + b(0) + c$$

$$\underline{0 = c}$$

(4,4)
x y

$$0 = a(4)^2 + b(4) + c$$

$$0 = 16a + 4b + c$$

$$0 = 49a - 7b$$

$$0 = 16a + 4b$$

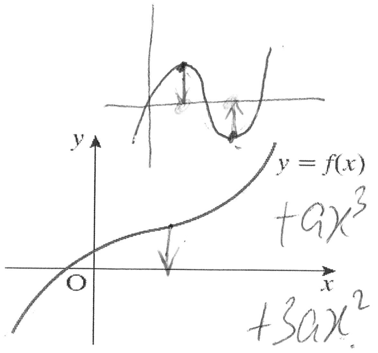
Solve simultaneous equations

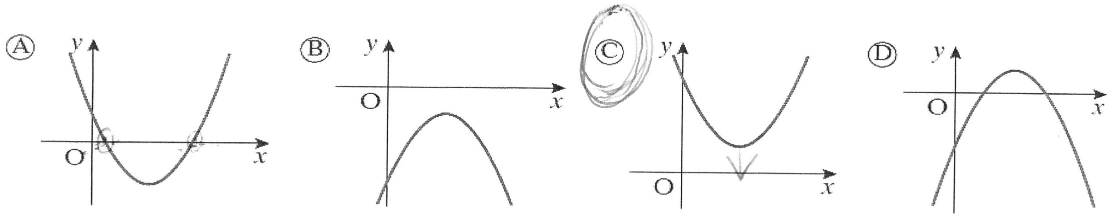
Differentiation

The graph of a cubic function $y = f(x)$ is shown on the right.

One of the four diagrams A, B, C, D below shows the graph of the derivative of f.

State which one it is, and justify your answer.





Since $f(x)$ has a point of inflection where graph C has its turning point.

Gas is escaping from a spherical balloon at the rate of $10\pi\,\text{cm}^3$ per minute. How fast is the radius decreasing when the radius is 5 cm?

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = 10\pi \qquad \text{find } \frac{dr}{dt}$$

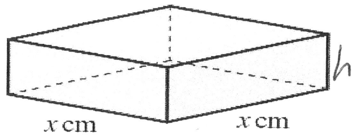
$$\frac{dV}{dr} = (3)\frac{4}{3}\pi r^2 = 4\pi r^2$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times 10\pi$$

$$= \frac{10\pi}{4\pi r^2} = \frac{5}{2r^2}$$

when $r = 5$, $\frac{5}{2(5)^2} = \frac{5}{50} = \frac{1}{10}$ cm/min

The diagram shows a rectangular cake-box, with no top, which is made from thin card. The volume of the box is 500 cm^3. The base of the box is a square with sides of length x cm.



(i) Show that the area, A cm^2, of card used to make the box is given by $A = x^2 + \frac{2000}{x}$.

(ii) Find the minimum area of card used.

$V = l \times w \times h$

$500 = x \times x \times h$

$500 = x^2 \times h$

$\frac{500}{x^2} = h$

Base: $x \times x$

Sides: $x \times h \Rightarrow$ 4 sides

Total: $x^2 + 4 \times (x \times h)$

Area $= x^2 + 4xh$

$A = x^2 + 4x\left(\frac{500}{x^2}\right)$

$A = x^2 + \frac{2000}{x}$

Note: $\frac{2000}{x} = 2000\left(\frac{1}{x}\right)$

$= 2000x^{-1}$

(ii) $\frac{dA}{dx} = 2x + (-1)2000x^{-2}$

$2x - \frac{2000}{x^2} = 0$ Solve!

$(x^2)(2x) - (x^2)\left(\frac{2000}{x^2}\right) = 0(x^2)$

$2x^3 - 2000 = 0$

$(\div 2)$ $x^3 - 1000 = 0$

$x^3 = 1000$

$x = \sqrt[3]{1000}$

$x = 10$

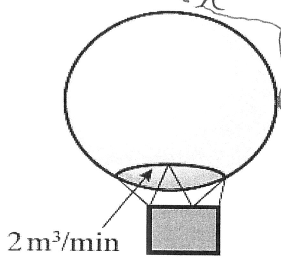
So when $x = 10$, $A = x^2 + \frac{2000}{x}$

$A = (10)^2 + \frac{2000}{(10)} = 300$ cm^2

A spherical hot-air balloon is being blown up so that its volume increases at a constant rate of 2 m^3 per minute.

(i) Find the rate of increase of the radius when $r = 2.5$ m.

(ii) Find the rate of increase of its surface area when $r = 2.5$ m.



$\frac{dV}{dt} = 2$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = (3)\frac{4}{3}\pi r^2 = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times 2$

find $\frac{dr}{dt}$ when $r = 2.5$

$\frac{2}{4\pi r^2} = \frac{1}{2\pi r^2} = \frac{dr}{dt}$

When $r = 2.5$, $\frac{1}{2\pi(2.5)^2} = \frac{2}{25\pi}$ metres per min.

(ii) $A = 4\pi r^2$

$\frac{dA}{dr} = (2)4\pi r = 8\pi r$

find $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\frac{dA}{dt} = 8\pi r \times \frac{1}{2\pi r^2} = \frac{8\pi r}{2\pi r^2} = \frac{4}{r}$

When $r = 2.5$, $\frac{4}{2.5} = 1.6$ m^2/min.

A particle moves in a straight line so that, after a time of t seconds, its distance s metres from a fixed point O is given by $s = 2t^3 - 24t$.

(i) Find the speed of the particle after 4 seconds.

(ii) After how many seconds is the particle at rest? Speed is zero

(i) $S = 2t^3 - 24t$

$\frac{ds}{dt} = (3)2t^2 - 24$

$6t^2 - 24$

When $t=4$ $6(4)^2 - 24$

72 metres per second

(ii) $S = 2t^3 - 24t$

$\frac{ds}{dt} = (3)2t^2 - 24$

$6t^2 - 24 = 0$

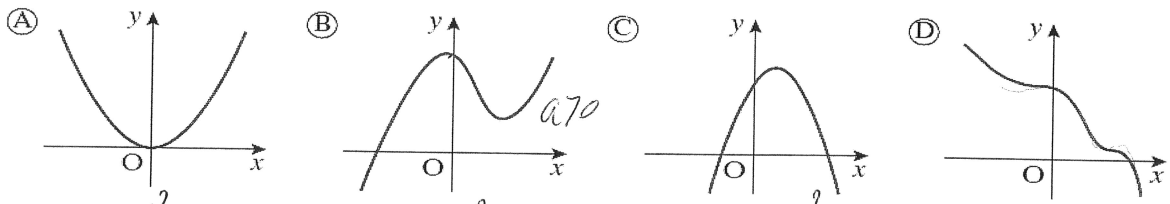
$\frac{6t^2}{6} = \frac{24}{6}$

$t^2 = 4$

$t = \pm 2$

$t = -2$ N/A $t = 2$ $\boxed{2 \text{ secs}}$

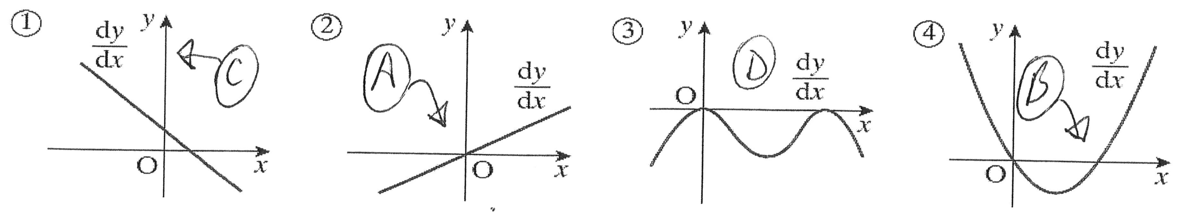
The graphs of four functions A, B, C and D are given below:



ax^2 ax^3 $-ax^2$

$2ax$ $3ax^2$ $-2ax$

The graphs of the slope functions of these four functions are shown below.
Match each graph to the graph of its slope function.



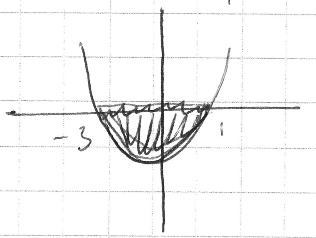
Find the range of values of x for which the function $f(x) = x^3 + 3x^2 - 9x$ is decreasing.

$f'(x) < 0$.

$f'(x) = 3x^2 + 6x - 9$

$3x^2 + 6x - 9 < 0$

$\div 3$ $\quad x^2 + 2x - 3 < 0$



Consider: $x^2 + 2x - 3 = 0$

$(x-1)(x+3) = 0$

Roots are: $x = 1 \quad x = -3$

$x > -3$ and $x < 1$.

The curve $f(x) = x^3 - bx^2 - 9x + 7$ has a stationary point when $x = -1$.
Find the value of b.

$f'(x) = 0$

$f'(x) = 3x^2 - 2bx - 9$

When $x = -1$, $\quad 3(-1)^2 - 2b(-1) - 9 = 0$

$3(1) + 2b - 9 = 0$

$2b - 6 = 0$

$2b = 6$

$b = 3$

Find the coordinates of the two points on the curve $y = x^3 - 3x^2 - 5x + 10$ where the tangents to the curve are parallel to the line $y = 4x - 7$.

slope!

$M = 4$

$\frac{dy}{dx} = 3x^2 - 6x - 5$

$3x^2 - 6x - 5 = 4$

$3x^2 - 6x - 9 = 0$

$\div 3$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$x = -1 \quad x = 3$

When $x = -1$

$y = (-1)^3 - 3(-1)^2 - 5(-1) + 10$

$y = 11 \quad (-1, 11)$

When $x = 3$

$y = (3)^3 - 3(3)^2 - 5(3) + 10$

$y = -5 \quad (3, -5)$.

Differentiate each of these, expressing each answer in its simplest form:

(i) $y = \ln(3x^4)$

Chain Rule

(i) $\ln \square \rightarrow \frac{1}{\square} \cdot \blacksquare$

$\frac{dy}{dx} = \frac{1}{(3x^4)} \cdot (4)3x^3$

$\frac{12x^3}{3x^4} = \frac{4}{x}$ or $4x^{-1}$

(ii) $y = \ln\left(\frac{3}{\sqrt{x}}\right) \rightarrow$ use laws of Logs to simplify!

(ii) $\ln 3 - \ln \sqrt{x}$

$\ln 3 - \ln x^{\frac{1}{2}}$

$\ln 3 - \frac{1}{2} \ln x$ now differentiate!

this is just a number

$0 - \frac{1}{2}\left(\frac{1}{x}\right) \Rightarrow -\frac{1}{2x}$.

The equation of a curve is $y = x \sin 2x$.
Find the slope of the tangent to the curve at the point where $x = \frac{\pi}{3}$.

Chain Rule

$u = x \quad v = \sin 2x$

$\sin \square \to \cos \square . \square$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = (\cos(2x))(2)$

$du/dx = 1 \quad dv/dx = 2\cos 2x$

$(x)(2\cos 2x) + (\sin 2x)(1)$

$2x\cos 2x + \sin 2x$

$x = \pi/3 \quad 2(\pi/3)\cos(2(\pi/3)) + \sin(2(\pi/3))$

$2(\pi/3)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right) \Rightarrow -\pi/3 + \frac{\sqrt{3}}{2}$

Find $\frac{dy}{dx}$ for each of these:

(i) $y = 3x^2 - x + \frac{3}{x}$ (ii) $y = \frac{3x^2}{x-1}$ (iii) $y = \cos^2 4x$

(i) $y = 3x^2 - x + 3x^{-1}$

$\frac{dy}{dx} = 6x - 1 + (-1)3x^{-2}$

$6x - 1 - 3x^{-2}$

$6x - 1 - \frac{3}{x^2}$

(ii) $u = 3x^2 \quad v = x - 1$

$\frac{du}{dx} = 6x \quad \frac{dv}{dx} = 1$

$\frac{(x-1)(6x) - (3x^2)(1)}{(x-1)^2}$

$\frac{6x^2 - 6x - 3x^2}{(x-1)^2}$

$\frac{3x^2 - 6x}{(x-1)^2}$

(iii) $y = \cos^2 4x$ P.T.A.

$y = (\cos 4x)^2$

$\square^n \to n\square^{n-1} \square$

$2(\cos 4x)^1(-\sin(4x)(4))$

$\cos \square \to -\sin \square . \square$

$-8\cos 4x \sin 4x$.

Differentiate $y = x^2 + 3x - 4$ from first principles.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^2 + 3x - 4$$

$$f(x+h) = (x+h)^2 + 3(x+h) - 4$$

$$\begin{array}{rl} f(x+h) & \cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h - \cancel{4} \\ - f(x) & -x^2 \qquad\qquad\quad - 3x \qquad\quad + 4 \\ \hline & = 2xh + h^2 + 3h \end{array}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{2x\cancel{h}}{\cancel{h}} + \frac{h^{\cancel{2}}}{\cancel{h}} + \frac{3\cancel{h}}{\cancel{h}}$$

$$\frac{f(x+h) - f(x)}{h} = 2x + h + 3$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 2x + (0) + 3 = 2x + 3$$

Integration

Integrate each of these:

(i) $\int \sin 3x \, dx$ (ii) $\int \cos 5x \, dx$ (iii) $\int (2\sin x + 3\cos 2x)\, dx$

$-\frac{1}{3}\cos 3x + c$

$\frac{1}{5}\sin 5x$

$-2\cos x + 3\left(\frac{1}{2}\right)\sin 2x + c$

$-2\cos x + \frac{3}{2}\sin 2x + c$

The three numbers $n-2$, n and $n+3$ are the first three terms of a geometric sequence. Find the value of n and hence write down the first four terms of the sequence.

$n-2 \quad n \quad n+3$

$\times r \quad \times r$

$r = \frac{n}{n-2}$ also $r = \frac{n+3}{n}$

$\frac{n}{n-2} = \frac{n+3}{n}$

$(n)(n-2)\left(\frac{n}{n-2}\right) = (n)(n-2)\left(\frac{n+3}{n}\right)$

$n^2 = (n-2)(n+3)$

$n^2 = n^2 + 3n - 2n - 6$

$-n^2 \quad -n^2$

$0 = n - 6$

$6 = n$

$n-2$

$6-2 = 4 \quad T_1$

$n = 6 \quad T_2$

$n+3$

$6+3 = 9 \quad T_3$

$\frac{27}{2} \quad T_4$

$\times \frac{3}{2}$ $\times \frac{3}{2}$ $\times \frac{3}{2}$

Find each of these:

(i) $\int\left(\frac{x^3-2}{x^2}\right)\mathrm{d}x$ (ii) $\int(\sqrt{x}-3)\,\mathrm{d}x$ (iii) $\int(\sqrt{x}+3)^2\,\mathrm{d}x$

$\int \frac{x^3}{x^2} - \frac{2}{x^2}\,dx$

$\int x - 2\left(\frac{1}{x^2}\right)dx$

$\int x - 2x^{-2}\,dx$

$\int x^{\frac{1}{2}} - 3\,dx$

$\int(\sqrt{x}+3)(\sqrt{x}+3)\,dx$

$\int x + 3\sqrt{x} + 3\sqrt{x} + 9\,dx$

$\int x + 6\sqrt{x} + 9\,dx$

$\int x + 6x^{\frac{1}{2}} + 9\,dx$

Find each of these integrals:

(i) $\int e^{5x}\,\mathrm{d}x$

$e^{ax} \rightarrow \frac{1}{a}e^{ax}$

$\frac{1}{5}e^{5x} + C$

Evaluate each of these definite integrals:

(i) $\int_0^3 (e^{2x} + 1)\,dx$

$$\left.\frac{1}{2}e^{2x} + x\right]_0^3$$

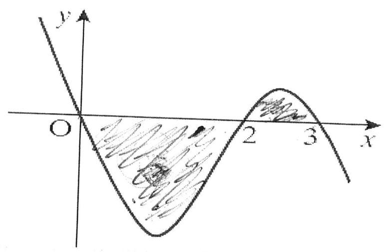
$$\left(\frac{1}{2}e^{2(3)} + (3)\right) - \left(\frac{1}{2}e^{2(0)} + (0)\right)$$

$$\left(\frac{1}{2}e^{6} + 3\right) - \left(\frac{1}{2}e^{0}\right)$$

$$\frac{1}{2}e^{6} + 3 - \frac{1}{2}$$

$$\frac{1}{2}e^{6} + \frac{5}{2} \checkmark$$

Find the exact area of the regions enclosed by the graph of $y = x(2 - x)(x - 3)$ and the x-axis.



$$\int_0^2 x(2-x)(x-3)\,dx + \int_2^3 x(2-x)(x-3)\,dx$$

$$\int_0^2 -x^3+5x^2-6x\,dx + \int_2^3 -x^3+5x^2-6x\,dx$$

$$\left.\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_0^2 + \left.\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_2^3$$

$$\left[\left(\frac{-(2)^4}{4} + \frac{5(2)^3}{3} - \frac{6(2)^2}{2}\right) - \left(\frac{-(0)^4}{4} + \frac{5(0)^3}{3} - \frac{6(0)^2}{2}\right)\right] + \left[\left(\frac{-(3)^4}{4} + \frac{5(3)^3}{3} - \frac{6(3)^2}{2}\right) - \left(-2\frac{2}{3}\right)\right]$$

$$\left[-2\frac{2}{3} - 0\right] + \left[-2\frac{1}{4} + 2\frac{2}{3}\right]$$

$$2\frac{2}{3} + \frac{5}{12} \longrightarrow \text{Total: } 3\frac{1}{12} \text{ sq. units}$$

A particle is moving in a straight line such that after t seconds, its velocity is v m/sec, where $v = 6t + 12t^2$.

Find (i) the average velocity during the first two seconds of motion
(ii) the average acceleration between $t = 1$ and $t = 5$.

Differentiation: Distance $\longrightarrow$ Speed $\longrightarrow$ Acceleration

integration   integration

$$\frac{1}{b-a}\int_a^b v\,dt$$

(i) $v = 6t + 12t^2$

$$\frac{1}{2-0}\int_0^2 6t + 12t^2\,dt$$

$$\frac{1}{2}\left[\frac{6t^2}{2} + \frac{12t^3}{3}\right]_0^2$$

$$\frac{1}{2}\left[\left(\frac{6(2)^2}{2} + \frac{12(2)^3}{3}\right) - \left(\frac{6(0)^2}{2} + \frac{12(0)^3}{3}\right)\right]$$

$$\frac{1}{2}\left[(12+32) - (0)\right] \rightarrow 22\text{ m/sec.}$$

$v = 6t + 12t^2$

acc. $= 6 + 24t$

(ii) $$\frac{1}{5-1}\int_1^5 6 + 24t\,dt$$

$$\frac{1}{4}\left[6t + \frac{24t^2}{2}\right]_1^5$$

$$\frac{1}{4}\left[\left(6(5) + 12(5)^2\right) - \left(6(1) + 12(1)^2\right)\right]$$

78 m/sec^2

If $f(x) = x\sin 2x$, find $f'(x)$.

Use your result to find $\int 2x\cos 2x\,dx$.

$f(x) = x\sin 2x$

$u = x \qquad v = \sin 2x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2\cos 2x$

$(x)(2\cos 2x) + (\sin 2x)(1)$

$2x\cos 2x + \sin 2x$

$$\int 2x\cos 2x + \sin 2x\,dx = x\sin 2x + c$$

$$\int 2x\cos 2x\,dx + \int \sin 2x\,dx = x\sin 2x + c$$

$$\int 2x\cos 2x\,dx = x\sin 2x - \int \sin 2x\,dx + c$$

$$= x\sin 2x + \frac{1}{2}\cos 2x + c.$$

Find the average value of the function $f(x) = 2x^2 - x$ over the interval $[0, 4]$.

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\frac{1}{4-0}\int_0^4 2x^2 - x\,dx$$

$$\frac{1}{4}\left[\frac{2x^3}{3} - \frac{x^2}{2}\right]_0^4$$

$$\frac{1}{4}\left[\left(\frac{2(4)^3}{3} - \frac{(4)^2}{2}\right) - \left(\frac{2(0)^3}{3} - \frac{(0)^2}{2}\right)\right]$$

$$\frac{1}{4}\left[\left(\frac{128}{3} - 8\right) - (0)\right] \longrightarrow 8\tfrac{2}{3}$$