Math 1497 Calc 2

So far we have considered functions

$$y = f(x)$$

ex $\quad y = \sin x, \quad y = e^x, \quad y = x^2+2x+4$

and in Calc 1, we developed concepts like limits, derivative, integral etc. We now wish to consider a larger class of equations called "parametric" equations. These are given by

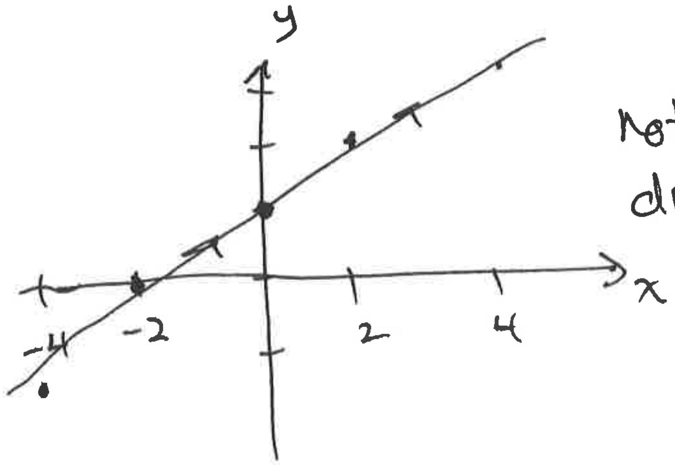
$$x = f(t), \quad y = g(t) \qquad t \text{- parameter}$$

ex $\quad x = 2t, \quad y = t+1, \quad t: -2 \to 2$

or $-2 \le t \le 2$

Table of values

t	x	y
-2	-4	-1
-1	-2	0
0	0	1
1	2	2
2	4	3



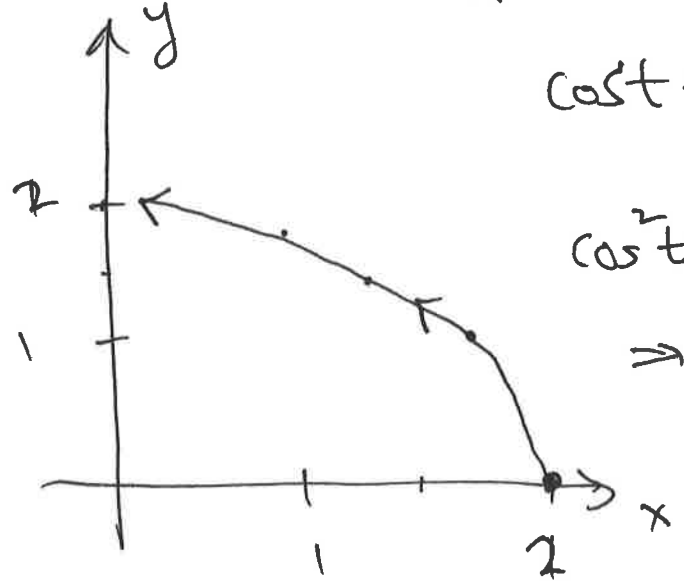
Note direction

we see that by eliminating t then $t = x/2$

so $y = \frac{x}{2} + 1$ – eqⁿ of a straight line

ex $x = 2\cos t, \; y = 2\sin t$

t	x	
0	2	0
$\frac{\pi}{6}$	$2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$	1
$\pi/4$	$2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$	$\sqrt{2} = 1.41$
$\pi/3$	$2 \cdot \frac{1}{2} = 1$	$\sqrt{3} = 1.73$
$\pi/2$	0	2



looks like a circle. In fact

$\cos t = \frac{x}{2} \quad \sin t = \frac{y}{2}$

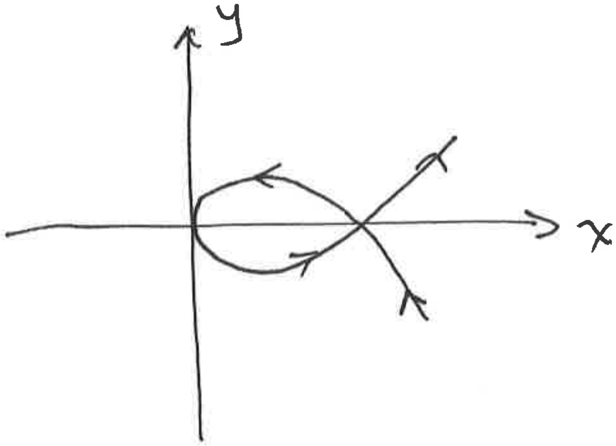
$\cos^2 t + \sin^2 t = 1$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{4} = 1$

$x^2 + y^2 = 4$ ✓

Some times even a table of value is still difficult to use and we use computer algebra packages

$x = t^2, \quad y = t^3 - t \qquad -2 \le t \le 2$



We see from the graph the curve crosses itself and it looks like at $(1,0)$. Let find the t values where this happen

so if $x = 1$ then $x = t^2 = 1 \Rightarrow t = \pm 1$

if $y = 0$ then $y = t^3 - t = 0 \Rightarrow t(t-1)(t+1) = 0$

so yes at $t = \pm 1$

the value at $t = 0$ is

when the curve is at $(0,0)$

we can eliminate t but we have $y = \pm(x\sqrt{x} - \sqrt{x})$

So can we calculate derivative of Parameter eq^n. For example, find the equation of the tangents at the pt of ~~[illegible]~~ intersection.

So we need $\frac{dy}{dx}$

If $x = f(t)$ & $y = g(t)$

then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$

So for previous ex

$x = t^2, \quad y = t^3 - t$

$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 3t^2 - 1$ & $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-1}{2t}$

So at the pt of intersection at $t = \pm 1$

$t = -1$

$\frac{dy}{dx} = \frac{3(-1)^2 - 1}{2(-1)} = -1$

$t = 1$

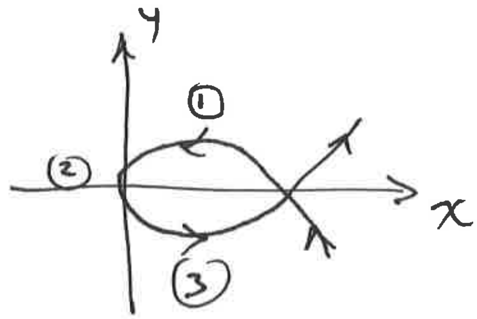
$\frac{dy}{dx} = \frac{3(1)^2 - 1}{2(1)} = 1$

The 2 tangents are

$$y - 0 = -1(x-1) \quad \& \quad y - 0 = 1(x-1)$$

we want to note a few things here

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2t}$$



at ① $t = \frac{-1}{\sqrt{3}}$

where $\frac{dy}{dx} = \frac{3\left(\frac{-1}{\sqrt{3}}\right)^2 - 1}{2\left(-\frac{1}{\sqrt{3}}\right)} = 0$ horizontal tangent

at ② $t = 0$

where $\frac{dy}{dx} = $ "$-\frac{1}{0}$" a vertical tangent

at ③ $t = \frac{1}{\sqrt{3}}$

$$\frac{dy}{dx} = \frac{3\left(\frac{1}{\sqrt{3}}\right)^2 - 1}{2\left(\frac{1}{\sqrt{3}}\right)} = 0$$ another horizontal tangent

so if $\frac{dx}{dt} = 0$ Vert. tang. $\frac{dy}{dt} = 0$ horz. tang.

Further Notes

2nd Derivatives

if $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ is a function of t

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$ Chain Rule

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dt}\Big/\frac{dx}{dt}\right)}{\frac{dx}{dt}}$$

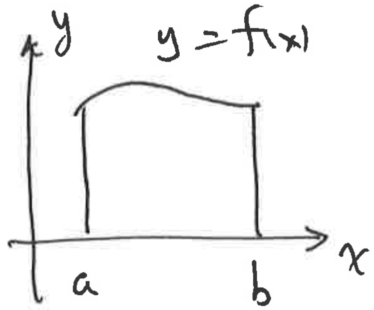
So previous ex

$x = t^2$, $y = t^3 - t$ $\quad \frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 3t^2 - 1$

& $\frac{dy}{dx} = \frac{3t^2-1}{2t} = \frac{3}{2}t - \frac{1}{2t}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3}{2}t - \frac{1}{2t}\right)}{2t} = \frac{\frac{3}{2} + \frac{1}{2t^2}}{2t} = \frac{3t^2+1}{4t^3}$$

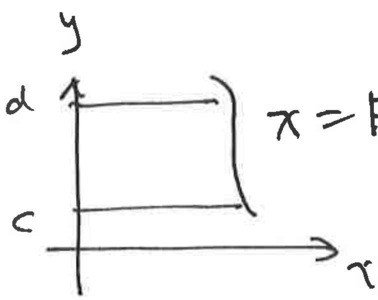
Areas



$\int_a^b f(x)\,dx$ ← this is y

so $\int_{t_1}^{t_2} y\,dx = \int_{t_1}^{t_2} g(t)\,f'(t)\,dt$ ∵ $x = f$, $y = g$

Also



$\int_c^d F(y)\,dy$

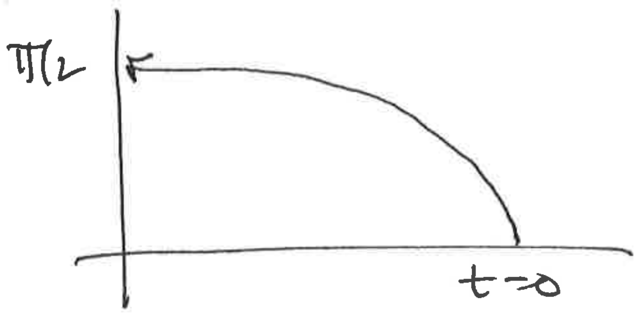
so $\int_{t_1}^{t_2} x\,dy = \int_{t_1}^{t_2} f(t)\,g'(t)\,dt$

important to note dx & dy must be positive so must the integrands.

Ex $x = a\cos t$, $y = b\sin t$ a, b #s

$\frac{x}{a} = \cos t$, $\frac{y}{b} = \sin t \Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

ellipse



$x = a\cos t \qquad dx = -a\sin t\,dt$

$y = b\sin t \qquad dy = b\cos t\,dt$

note $\quad dx < 0$

(1) $$\int_0^b y\,dx = \int_0^{\pi/2} b\sin t\,(-a\sin t\,dt = -ab\int_0^{\pi/2}\sin^2 t\,dt$$

$$= -ab\int_0^{\pi/2}\frac{1-\cos 2t}{2}\,dt = -ab\quad \frac{t}{2} - \frac{\sin 2t}{a}\Big|_0^{\pi/2}$$

$$= -\frac{\pi ab}{4}$$

(2) $$\int_a^b x\,dy = \int_0^{\pi/2} a\cos t\cdot b\cos t\,dt$$

$$= ab\int_0^{\pi/2}\cos^2 t\,dt$$

$$= ab\int_0^{\pi/2}\frac{1+\cos 2t}{2}$$

$$= ab\quad \frac{t}{2} + \frac{\sin 2t}{4}\Big|_0^{\pi/2} = \frac{ab\pi}{4}$$