Rayleigh Lomax Distribution

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Abstract: A three parameters composition of Rayleigh density and Lomax density is introduced in the study namely Rayleigh Lomax Distribution (RLD). The mathematical properties of the proposed model are studied which include mean, median, mode, generating functions, quantile functions, and others. The reliability analysis of the model is examined by means of various functions which include hazard, reverse hazard, cumulative hazard, and survival functions. In addition, the Order Statistic of the model is also discussed. The model is applied upon real data set to assess the usefulness and flexibility of the proposed model by the method of Maximum Likelihood.

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1. Introduction

There are various lifetime distributions such as Exponential, Rayleigh, Weibull, Pareto, Lomax and many others which are used for modeling purposes. These are popular models useful in the modeling of probability distribution and assess their effectiveness practically. The Rayleigh distribution was introduced by Rayleigh (1880) and is widely used in different field like engineering, sciences, failure rates etc. The Lomax distribution was proposed by Lomax (1954) and is widely used in medical, inequalities of wealth, business failures, actuarial sciences etc. Harris (1968) applied Lomax model for modeling wealth and income.

2. Method and Model Development

There are various methods available in the literature for development of probability distribution. The Rayleigh Lomax Distribution (RLD) is generated via the methodology of Al-Kadim & Boshi (2013).

$$G(x) = \int_{0}^{1/1-F_2(x)} f_1(\mathrm{x})dx$$

Where $f_1(x)$ and $F_2(x)$ respectively is probability density function(pdf) of base distribution and distribution function (cdf) of second distribution that is used to develop mixture model form. This approach has been applied by various practitioners which include Nasiru & Luguterah (2015), El-Bassiouny et al. (2015) and Shabbir et al. (2016).

By putting the expressions of $f_1(x)$ and $F_2(x)$ through simplification

$$G_{RL}(x) = 1 - e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}$$

By differentiation of equation, the RLD pdf obtained is

$$g_{RL}(x) = \frac{\alpha}{\sigma^2\beta}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha+1} e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}} \quad ; x \geq -\beta, \alpha > 0, \beta > 0, \sigma > 0.$$

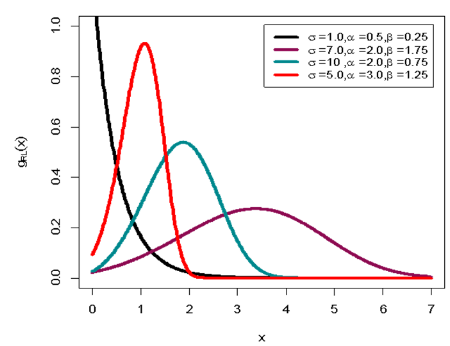

Figure 1. RLD pdf Plot for different values of α, β and σ.

3. RLD Properties and Reliability Analysis

3.1. Reliability Measures

The reliability measures of RLD are reliability function $[R(x)]$, failure rate function $[H(x)]$, reverse hazard function $[\tau(x)]$ and the cumulative hazard function $[H_x(x)]$. The expressions for all measures respectively are:

$$R_{RL}(x) = 1 - G_{RL}(x) = e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}$$

$$H_{RL}(x) = \frac{g_{RL}(x)}{1-G_{RL}(x)} = \frac{\frac{\alpha}{\sigma^2\beta}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha+1} e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}}{e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}}$$

$$\tau_{RL}(x) = \frac{g_{RL}(x)}{G_{RL}(x)} = \frac{\frac{\alpha}{\sigma^2\beta}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha+1} e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}}{1 - e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}}$$

$$H_{x-RL}(x) = -\ln[R_{RL}(x)] = \frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}$$

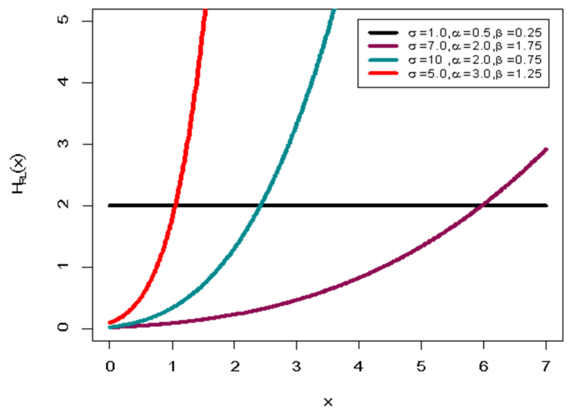

Figure 2. RLD Hazard Function Plot for different values of α, β and σ.

From Figure 2, it is observed that RLD has increasing hazard function and is also constant at α=0.5.

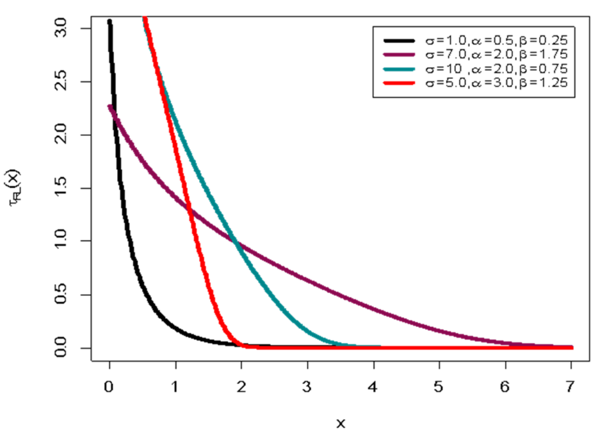

Figure 3. RLD Reverse Hazard Function Plot for different values of α, β and σ.

3.2. RLD Mathematical Properties

Several properties of RLD are derived which include moments, median, mode, generating function, mean deviation from mean and median, quantile function and random number generator. Expressions are respectively given as:

3.2.1. Moments

$$\mu'_r = \sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}(2\sigma^2)^{\frac{k}{2\alpha}}\beta^r\,\Gamma\left(\frac{r}{2\alpha}+1\right); r = 1,2,..$$

3.2.2. Median

Median can be obtained for RLD by using the expression

$$Median = \beta\left[\left(2\sigma^2 \ln(2)\right)^{1/2\alpha} - 1\right]$$

3.2.3. Mode

The mode is obtained by differentiating pdf of RLD with respect to x and then equating to zero. Thus, modethe is found at x_o by expression

$$x_o = \beta\left[\left(\frac{\sigma^2(1-2\alpha)}{\alpha}\right)^{1/2\alpha} - 1\right]$$

3.2.4. Generating function

The generating function of moments ($\boldsymbol{m.g.f.}$) of RLD is

$$M_x(t) = \sum_{v=0}^{+\infty}\sum_{k=0}^{c}\binom{c}{k}\frac{(t)^v(-1)^{c-k}(2\sigma^2)^{\frac{k}{2\alpha}}\beta^r}{c!} \times \Gamma\left(\frac{c}{2\alpha}+1\right)$$

3.2.5. Mean Deviation

The mean deviation for RLD is given as:

(i) Mean

$$D(\mu) = 2(\mu+\beta)\left[1 - e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{\mu+\beta}\right)^{-2\alpha}}\right] - 2\beta(2\sigma^2)^{\frac{1}{2\alpha}}\gamma\left(\frac{1}{2\alpha}+1, \frac{1}{2\sigma^2}\left(\frac{\beta}{\mu+\beta}\right)^{-2\alpha}\right)$$

(ii) Median

$$D(m) = \mu - 2\beta(2\sigma^2)^{\frac{1}{2\alpha}}\gamma\left(\frac{1}{2\alpha}+1, \frac{1}{2\sigma^2}\left(\frac{\beta}{m+\beta}\right)^{-2\alpha}\right) + 2\beta\left[1 - e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{m+\beta}\right)^{-2\alpha}}\right]$$

3.2.6. Quantile Function

RLD quantile function is given by

$$Q(s) = \beta\left[\left(2\sigma^2 \ln\left(\frac{1}{1-s}\right)\right)^{\frac{1}{2\alpha}} - 1\right]$$

3.2.7. Random Number Generator

Let U denote a uniform variate on interval 0 and 1. Then the random number generator for RLD is

$$x = \beta\left[\left(2\sigma^2 \ln\left(\frac{1}{1-U}\right)\right)^{1/2\alpha} - 1\right]$$

4. Order Statistics

The distribution of s^{th} order statistic of RLD is

$$g_{X(s)}(x) = \frac{n!}{(s-1)!\,(n-s)!}\,g(x)[G(x)]^{s-1}[1 - G(x)]^{n-s} \quad ; s = 1,2,3,\ldots$$

$$g_{X(s)}(x) = \frac{n!}{(s-1)!\,(n-s)!}\frac{\alpha}{\sigma^2\beta}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha+1} e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}\left[1 - e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}\right]^{s-1} \times \left[e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}\right]^{n-s} \quad ; s = 1,2,3$$

Then the distribution of smallest order statistic of RLD is

$$g_{X(1)}(x) = n\frac{\alpha}{\sigma^2\beta}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha+1} e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}\left[e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}\right]^{n-1}$$

Then the distribution of the largest order statistic of RLD is

$$g_{X(n)}(x) = n\frac{\alpha}{\sigma^2\beta}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha+1} e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}\left[1 - e^{-\frac{1}{2\sigma^2}\left(\frac{\beta}{x+\beta}\right)^{-2\alpha}}\right]^{n-1}$$

5. Maximum Likelihood Estimation

The maximum likelihood method is utilized for estimating parameters of RLD. The Log-Likelihood function is given by

$$lnL = n\,ln(\alpha) - 2n\,ln(\sigma) - nl\,n(\beta) + (1-2\alpha)\sum_{i=1}^{n} ln\left(\frac{\beta}{x_i+\beta}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-2\alpha}$$

Now differentiate log-likelihood with respect to σ, α and β respectively we obtain

$$\frac{\partial lnL}{\partial\sigma} = -\frac{2n}{\sigma} + ln(\beta) + \frac{1}{\sigma^3}\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-2\alpha}$$

$$\frac{\partial lnL}{\partial\alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n} ln\left(\frac{\beta}{x_i+\beta}\right) + \frac{1}{\sigma^2\beta^{2\alpha}}\sum_{i=1}^{n}(x_i+\beta)^{2\alpha}\,ln\left(\frac{\beta}{x_i+\beta}\right)$$

$$\frac{\partial lnL}{\partial\beta} = -\frac{2n\alpha}{\beta} - (1-2\alpha)\sum_{i=1}^{n}(x_i+\beta)^{-1} - \sum_{i=1}^{n}\frac{x\alpha}{\sigma^2}\left(\frac{\beta}{x_i+\beta}\right)^{-2\alpha}$$

The estimates of maximum likelihood are obtained via solving *above* equations by non-linear equation system. The iterative method developed by Newton Raphson is used to solve the equations. According to the large sample approximation, the estimators of the maximum likelihood approach can be treated as multivariate normal approximately. Therefore, when $n \to \infty$, the asymptotic distribution of $\hat{\sigma}, \hat{\alpha}$ and $\hat{\beta}$ is as follows

$$\begin{pmatrix}\hat{\sigma}\\ \hat{\alpha}\\ \beta\end{pmatrix} \to MVN\left[\begin{pmatrix}\sigma\\ \alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33}\end{pmatrix}\right]$$

Where $\hat{V}_{ij} = V_{ij}|_{(\hat{\sigma},\hat{\alpha},\hat{\beta})}$ the approximate Variance-Covariance Matrix which can be approximately determined by using the Information Matrix

$$V = \begin{pmatrix}V_{11} & V_{12} & V_{13}\\ V_{21} & V_{22} & V_{23}\\ V_{31} & V_{32} & V_{33}\end{pmatrix} = \begin{pmatrix}I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{pmatrix}^{-1}$$

Where:

$I = \begin{pmatrix}I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{pmatrix}$, is the Information Matrix.

For RLD, all the second order derivatives exist. Hence, we can obtain

$$I_{11} = \frac{\partial^2 lnL}{\partial\sigma^2},\quad I_{12} = I_{21} = \frac{\partial lnL}{\partial\sigma\partial\alpha}$$

$$I_{22} = \frac{\partial^2 lnL}{\partial\alpha^2},\quad I_{13} = I_{31} = \frac{\partial lnL}{\partial\sigma\partial\beta}$$

$$I_{33} = \frac{\partial^2 lnL}{\partial\beta^2},\quad I_{23} = I_{32} = \frac{\partial lnL}{\partial\alpha\partial\beta}$$

By solving the Fisher Matrix, we find the variance-covariance matrix. Hence, an approximate $100\,(1-\vartheta)\%$ confidence interval for σ, α and β is given by

$$\hat{\sigma} \pm z_{\frac{\vartheta}{2}}\sqrt{\hat{V}_{11}} \quad \hat{\alpha} \pm z_{\frac{\vartheta}{2}}\sqrt{\hat{V}_{22}} \quad \hat{\beta} \pm z_{\frac{\vartheta}{2}}\sqrt{\hat{V}_{33}}$$

Where $z_{\frac{\vartheta}{2}}$, is the upper $\frac{\vartheta}{2}$ percentile of the the standard normal distribution.

6. Application

In this section, RLD is employed on real data set *Cordeiro* et al. (2014). The data set consists of seventy-two observations as: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54,54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

We compare RLD with the following densities:

(i) Exponential Density

$$g(x) = \lambda e^{-\lambda x} \quad ; x > 0 \quad \lambda > 0.$$

(ii) Lomax Density

$$g(x) = \frac{\lambda\theta}{(1+\lambda x)^{\theta+1}} \quad ; x > 0, \quad \lambda, \theta > 0.$$

(iii) Rayleigh Density

$$g(x) = \frac{x}{\sigma^2}e^{x^2/2\sigma^2} \quad ; x > 0 \quad \sigma > 0.$$

The Maximum Likelihood estimates of four distributions are computed. The four criteria used for comparison are

a) AIC= -2lnL+2p
b) BIC=-2lnL+plog(n)
c) CAIC=-2lnL+2pn/(n-p-1)

d) Kolmogorov-Smirnov (K-S) Test

Where p denotes the number of model parameters and n is the total number of observations.

Table 1: *Maximum Likelihood Estimates and K-S Statistic*

Model	MLE	K-S
Exponential	$\hat{\lambda}$ =0.01003	0.2122
Rayleigh	$\hat{\sigma}$ = 90.69947	0.2873
Lomax	$\hat{\lambda}$ = 0.00258 $\hat{\theta}$ = 4.38206	0.2299
Rayleigh Lomax	$\hat{\sigma}$ =34.6320 $\hat{\alpha}$ =0.69165 $\hat{\beta}$ =0.39824	0.1440

Table 2: *Goodness of Fit criterions on the data set*

Model	AIC	BIC	CAIC	HQIC
Exponential	808.8	811.2	808.9	809.8
Rayleigh	818.6	820.8	818.6	819.5
Lomax	817.9	822.5	818.1	819.8
Rayleigh Lomax	800.6	807.4	800.9	803.3

From Table 1 it is observed that goodness of fit is minimum for RLD and from Table 2 it is observed that the minimum value of the measures of model criterions is produced by RLD from four criterions among models under evaluation.

7. Conclusion

In this study, a new probability model has proposed namely Rayleigh Lomax Distribution (RLD). The proposed model is found to be more flexible from the parent probability models. We have derived the structural properties of the distribution along with reliability analysis. Some order statistic expressions have also been derived. The new proposed model applied upon real data set provides a better fit than other considered models in the study**.**

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References :

1. Al-Kadim, K. A., & Boshi, M. A. (2013). Exponential Pareto Distribution. Mathematical Theory and Modeling, 3(5), pp. 135-146.
2. Cordeiro, G. M., Lemonte, A. J., & Ortega, E. M. (2014). The Marshall–Olkin family of distributions: mathematical properties and new models. *Journal of Statistical Theory and Practice*, *8*(2), 343-366.
3. El-Bassiouny, A. H., Abdo, N. F., & Shahen, H. S. (2015). Exponential Lomax distribution. *International Journal of Computer Applications*, *121*(13), pp. 24-29.
4. Harris, C. M. (1968). The Pareto distribution as a queue service discipline. *Operations Research*, *16*(2), 307-313.
5. Lomax, K. S. (1954). Business failures: Another example of the analysis of failure data. *Journal of the American Statistical Association*, *49*(268), 847-852.
6. Nasiru, S., & Luguterah, A. (2015). The new Weibull-Pareto distribution. *Pakistan Journal of Statistics and Operation Research*, *11*(1), 103-114.
7. Rayleigh, L. (1880). XII. On the resultant of a large number of vibrations of the same pitch and of arbitrary phase. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, *10*(60), 73-78.
8. Shabbir, M., Noor, N., Riaz, A., & Gull, H. (2016). The New Weibull Lomax Distribution. *Imperial Journal of Interdisciplinary Research*, *3*(1), pp. 1881-1885.

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