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Machian Inertia and the Isotropic Universe

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This paper addresses the origin of the forces of inertia. It proposes a Newton-Mach particle interaction force between all pairs of particles that depends on their relative acceleration and is proportional to the gravitational force between them. The motion of all objects therefore becomes directly influenced by all of the matter in the universe, as prescribed by Mach's principle. The effect of the observed hierarchical structure of the universe is considered and is used to ensure that the inertial force on an object is finite and isotropic. The instantaneous matter interaction force is justified and both Einstein's and Mach's objections to a Newtonian framework are discussed and shown to be absorbed by the proposed universal law of inertia.

KEY WORDS: Newton-Mach Paradigm; Cosmology.

1. NEWTON-MACH PARADIGM

Any Machian theory of inertia depends on instantaneous action at a distance, or as one might prefer to call it, mutual simultaneous far-actions. The reason for this is the requirement of simultaneous universal momentum and energy conservation which is well known from experiment and is the heart of Newtonian mechanics. To illustrate this point we consider the simple example of the falling apple to which Figure 1 refers. This diagram complies with d'Alembert's principle of Newtonian mechanics according to which all forces on a finite body or particle are in dynamic equilibrium at any instant. It is equivalent to saying their vector sum is zero. In Figure 1, IMD stands for an isotropic mass distribution. With M being the mass of the earth, m, the mass of the apple, G, Newton's constant of gravitation and $r_{a,e}$, the distance between the centres of gravity of the two objects, then Newton's

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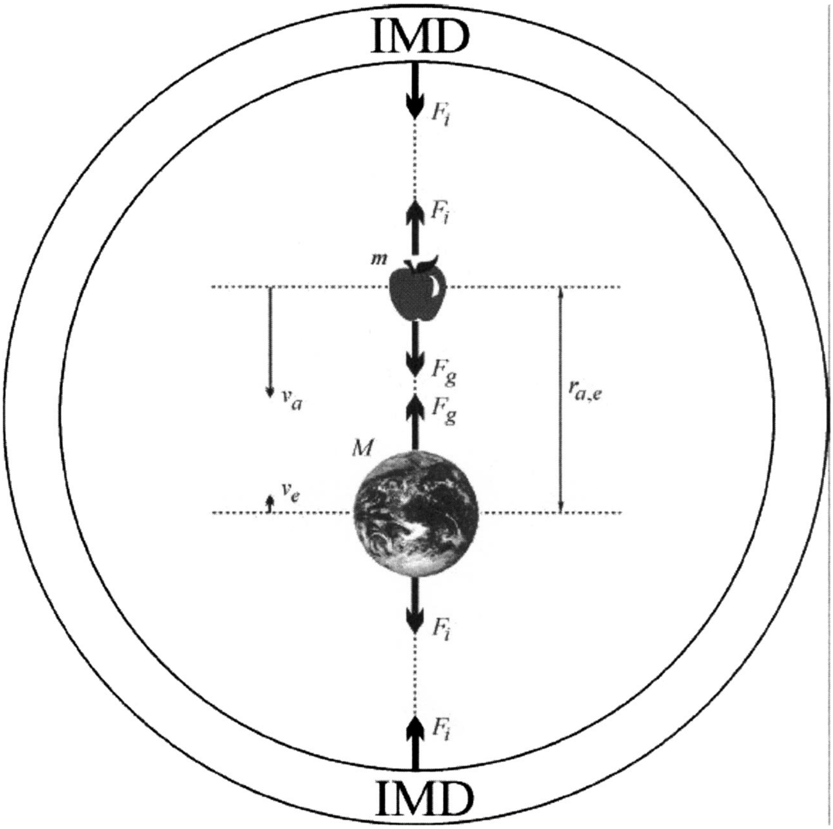

Figure 1. The fall of an apple, demonstrating instantaneous momentum conservation in accordance with D'Alembert's principle.

universal law of gravitation gives the mutual force of attraction between the apple and the earth as

$$F_{g(a,e)} = -G\frac{Mm}{r_{a,e}^2}. \tag{1}$$

The force is always negative, implying attraction. Further, assuming no external forces such as air resistance, at every instant, the downward velocity of the apple, v_a, and the associated upward velocity, v_e, of the earth must ensure momentum conservation. Therefore even while both are accelerating toward each other

$$Mv_e + mv_a = \text{Constant}. \tag{2}$$

The velocities cannot be referred to the frame of the earth because v_e would then be zero and momentum could not be conserved as the bodies accelerate toward each other. Mach (1960, p. 286) insisted that the two velocities have to be assessed relative to the fixed stars. In this paper it will be argued that Mach's unique inertial reference frame is more sensibly taken to be an isotropic distribution of matter which for our purposes may be treated as being at rest with respect to our galaxy.

The potential energy, $P_{a,e}$, of Newton's universal gravitation for the apple-earth combination is the energy stored when the earth and apple centres of gravity were moved apart from 0 to $r_{a,e}$ against the force of gravity,

$$P_{a,e} = \int_0^{r_{a,e}} -F_{g(a,e)}\, dr. \tag{3}$$

In Newtonian dynamics, the negative gradient of the gravitational potential function defines the mutual gravitational attraction, or

$$F_{g(a,e)} = -\frac{dP_{a,e}}{dr}. \tag{4}$$

In order to maintain instantaneous energy conservation, the loss in potential energy must at all times be equal to the gain in kinetic energy when the velocities are expressed relative to the Machian frame of inertia.

There is little doubt that kinetic energy must reside in the moving body which possesses it, however the location of the storage of potential energy is not so obvious. In non-Newtonian field theories, the stored potential energy is a physical commodity which resides in the field surrounding the mobile bodies. If this were correct, then the conversion of potential to kinetic energy would take travel time and it would be impossible to conserve energy instantaneously.

In strictly Newtonian physics, energy is always associated with matter. It is then logical to assume that the potential energy of gravitation is simply a mathematical representation of distant matter force interactions. As well, the principles of momentum and energy conservation require the forces of attraction, $F_{g(a,e)}$, (Figure 1) to act simultaneously on the apple and the earth. Consequently, the experimentally well established concepts of both momentum and energy conservation provide compelling support for the concept of instantaneous action at a distance.

Figure 1 also shows the forces of inertia, $\vec{F}_i$, which Newton defined as being equal and opposite to the external force causing the observed acceleration, $\vec{F}_e$, that is

$$\vec{F}_i = -\vec{F}_e = -m\vec{a}, \tag{5}$$

where $\vec{a}$ is the acceleration of m relative to Newton's proposed absolute space. Mach, however, insisted that $\vec{a}$ is the acceleration relative to the fixed stars, which in the present analysis, is taken to be equal to the acceleration relative to the isotropic mass distribution, (IMD).

We now adhere to the Newtonian view that all fundamental forces of nature are attractions or repulsions between two entities of matter (Graneau 1999). This becomes the most generally valid form of Newton's third law. As a result, one must discover what particles are interacting with an accelerating object in order to create the inertia force. These interacting particles must form an isotropic distribution as the magnitude of the forces of inertia are independent of the direction of the externally applied force. It is therefore proposed that the cause of and the reaction to the inertia forces is distributed over an IMD, scattered throughout the universe.

The inertia force $\vec{F}_i$ and its equal and opposite reaction force on the IMD can be treated as having a line of action, as shown in Figure 1, which is co-linear with the force $\vec{F}_g$ on the apple. Since the earth is accelerating upward, it will also be subject to a force of inertia equal and opposite to $\vec{F}_g$. This leads to a second reaction force, $\vec{F}_i$, on the IMD. Hence we have to consider three attractions: (1) apple-earth, (2) apple-IMD and (3) earth-IMD.

Mach criticized much of Newton's wording of the latter's theory of dynamics. He reserved the strongest objection for Newton's concepts of absolute space and absolute time. In the preface to the seventh (German) edition (1912) of his book, *The Science of Mechanics*, Mach (1960, p. xxviii), wrote (in English translation):

> "With respect to the monstrous conception of absolute space and absolute time I can retract nothing. Here I have only shown more clearly than hitherto that Newton indeed spoke much about these things, but throughout made no serious application of it."

The mechanically expressed fundamental laws of Newtonian mechanics are still correct and used daily, although most scientists have agreed with Mach regarding the unreality of absolute space and time. The implication is that the force of inertia, $\vec{F}_i$, on the apple of Fig. 1 is not a local interaction with absolute space, but is the consequence of a vast number of remote interactions with all of the matter in the universe. The interactions that significantly determine the magnitude and direction of the inertia force are those that involve the vast isotropic matter distribution of the distant universe. This philosophical change has no effect on the equations of Newtonian dynamics and the magnitude of the force of inertia is still given by Newton's second law of motion, Eq. (5). Mach (1960, p. 287) developed an argument which concludes:

> "... we see that even in the simplest case, in which apparently we deal with the mutual action of only two masses [apple and earth], the neglecting of the rest of the world is impossible."

This last statement comes nearest to what is now generally referred to as *Mach's Principle*. Einstein (1920, p. 71) accepted that Mach had corrected one of the two perceived fundamental flaws of Newtonian mechanics, and thus he sought to incorporate Mach's principle into his own relativity theory. The complete paradigm suggested by this principle however still requires a law of nature which describes the inertia force interaction between a particle in the laboratory and another particle in the distant universe. We will call this the Machian inertial particle interaction law.

2. PREVIOUS ATTEMPTS TO DISCOVER THE MACHIAN INTERACTION LAW

Five serious attempts have been made in the second half of the 20th century to discover the Machian interaction law that could explain inertia. The first was due to Sciama (1953). He argued that matter had inertia only in the presence of other matter. In other words, inertia in a particle was induced by other remote particles. He called upon an analogy with electromagnetic induction. This became the pattern followed by all five previous investigators of the Machian particle interaction law.

Eighteen years after Sciama, French (1971, p. 542) derived an inertia induction law in his textbook, *Newtonian Mechanics*. He called it a speculation on the origin of inertia. Apparently unaware of Sciama's efforts, French also relied on the electromagnetic analogy. On the basis of Mach's principle, he argued that the linear inertia force ($\vec{F}_i$ in Figure 1) and defined by Eq. (5) as $-m\vec{a}$, must be ascribable to the acceleration of other bodies in the universe relative to a particle on earth. This implied a mutual simultaneous interaction of widely separated particles and bodies in a manner comparable to Newton's universal theory of gravitation but in a manner that also depended on relative acceleration.

To discover the origin of inertia, French used the electromagnetic analogy depicted in Figure 2. Two electric charges, $+q_1$ and $-q_2$, attract each other according to Coulomb's law by the force

$$F_{c(1,2)} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2 a}{r^2}, \tag{6}$$

where r is the distance between the charges and $(1/4\pi\varepsilon_0)$ is a dimensional constant. $F_{c(1,2)}$ is a ponderomotive (mechanical) force and it obeys Newton's third law. French proposes that q_2 be given an acceleration, a, relative to q_1 caused by an external force, F_e. Therefore at any subsequent instant, q_2 is moving with a velocity, v, relative to q_1. The latter he assumes is stationary in the laboratory.

French then calculates the electrodynamic interaction of the two charges in motion relative to each other. The magnetic vector potential of the current element, $q_2 v$, at the position of q_1 is $(q_2 v/r)$ in the direction of the relative acceleration, a. In relativistic electromagnetism, the rate of change of the vector potential,

$$\frac{d}{dt}\frac{q_2 v}{r} = \frac{q_2\, a}{r} = E, \tag{7}$$

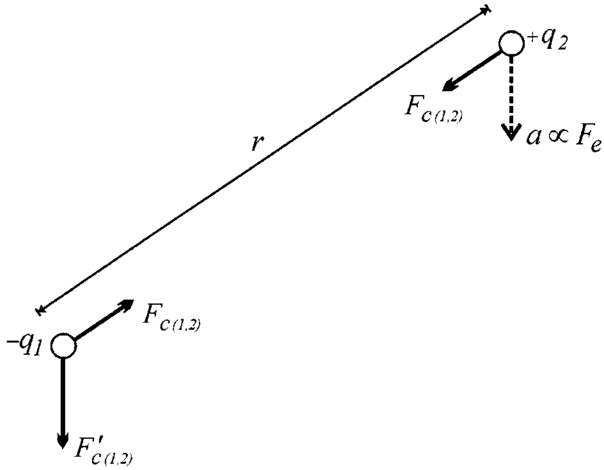

Figure 2. French's (1971) interaction of two electric charges.

results in an electric field strength, E, which then exerts an electromotive (not mechanical) force, $F'_{c(1,2)}$ on q_1 (see Figure 2). From this follows French's equation

$$F'_{c(1,2)} = \frac{1}{4\pi\varepsilon_0}\frac{q_1\, q_2\, a}{c^2\, r}. \tag{8}$$

The speed of light, c, has entered Eq. (8) as a consequence of the conversion from electrostatic units of charge to electrodynamic units of charge. This was in fact the context in which the constant, c, was first introduced into physics by Weber (1848) in his attempts to unify the existing action at a distance laws of electrostatics and electrodynamics. Consequently the charges in Eq. (8) are no longer the electrostatic charges expressed in Coulomb's law, Eq. (6).

It will be noted that from Eq. (6),

$$F'_{c(1,2)} = F_{c(1,2)}\frac{r\, a}{c^2}. \tag{9}$$

In French's speculation about the origin of inertia, Coulomb's law is taken as an analogy of Newton's law of universal gravitation. In order to achieve this, he substitutes two masses m and M for the two charges of Eq. (6), and the dimensional constant is replaced by Newton's Gravitational constant, G. From this he speculates that the same substitutions will also be valid in Eq. (8) yielding

$$F'_{i(1,2)} = G\frac{M\, ma}{c^2\, r}. \tag{10}$$

Consequently, he proposes that the total inertial force could be calculated if all of the objects in the universe acquire an acceleration, a, with respect to the mass, m. By summing over all masses except m, the inertial force on it can be expressed as

$$F_{inertial} = ma \sum_{all\ masses} \frac{G\,M}{c^2\,r}. \tag{11}$$

In order to agree with Newton's well known second law of motion,

$$\sum_{universe} \frac{G\,M}{c^2\,r} = 1. \tag{12}$$

Relying on figures which have at times been quoted for the radius of a spherical cosmos and the total mass contained in it, French claimed that Eq. (12) was not unreasonable. A controversial feature of French's theory, however is that the velocity of light, a fundamentally electrodynamic quantity, now enters the Newtonian dynamics of forces of gravitation and inertia in which it has no obvious meaning. As mentioned earlier, it appeared because French used two dimensionally differing types of charge in his electrical analogy whereas there is only one type of mass. From this, it can be concluded that the electrodynamic analogy is an artefact and French's Eq. (10), which he considered to be the Machian particle interaction law, is probably incompatible with Newtonian mechanics.

Three more attempts were made to discover the Machian particle interaction law which must underlie Newton's force of inertia, Eq. (5). These investigations were carried out by Burniston Brown (1982, chap. 7), Assis (1989) and Ghosh (2000, chap. 3). Although they all arrived at the same result as French, the latter authors provided more qualitative discussion on the nature of the universe. They agreed on the following premises:

- The Machian particle interaction is based on an action at a distance mechanism
- The observable universe is a sphere of finite radius with the Milky Way at its centre.
- There exists much isotropically distributed matter in the universe outside our home galaxy. This matter is responsible for the isotropic forces of inertia observed on earth. Burniston Brown includes in this all visible matter while Assis and Ghosh speak of an isotropic matter distribution superimposed on an anisotropic distribution.
- On earth, we experience local gravitational attractions described correctly by Newton's universal law of gravitation. This involves primarily the bodies of the solar system. As a consequence of the apparent isotropy of the extra-galactic cosmos, its gravitational effect cannot be measured. The observable Newtonian gravitational attractions involve so little matter that their anisotropic contribution to inertia forces is negligible.

Burniston Brown discussed retarded action at a distance, while Assis utilised Weber's instantaneous action at a distance. Ghosh mixed instantaneous with retarded action at a distance. They all, however, arrived at French's result of Eq. (10). This is due to the fact that Brown's force calculations ignore the retardation aspect, presumably because it became unmanageable. French's electrodynamic formula, Eq. (8), was derived with the help of relativistic field theory, while Burniston Brown and Assis relied on Weberian electrodynamics which did not contain fields. This is very surprising and suggests that special relativity, and field theory in general, is to some extent contained in Weber's electrodynamics. While Burniston Brown and Assis argue that their forces of inertia are of Newtonian gravitational origin, this cannot be true because Eq. (10) is not an inverse square law and it contains the velocity of light. None of these five authors addressed the issue of how their equation could lead to a finite and measurable force of inertia in a possibly infinite universe.

3. PROPOSED MACHIAN PARTICLE INTERACTION LAW

Accepting the Newtonian principle of inertia, which states that the force of inertia counteracts acceleration, we expect that a particle which accelerates in the midst of an isotropic mass distribution (IMD), in any arbitrary direction, will experience a repulsion from half the distribution in front of it and an attraction from the other half behind it. These repulsions and attractions must combine to create the measurable force of inertial resistance to acceleration as quantified by Newton's principle of inertia as expressed in Eq. (5). Further we never detect a velocity dependent Newtonian force of attraction or repulsion as expressed in Newton's first law and the principle of Galilean invariance. Therefore we only need to consider an interaction which is a function of relative position and acceleration.

We will now hypothesize the Machian particle interaction with distant matter on the basis of Eq. (5) without calling upon an electrodynamic analogy. We feel justified to utilise an instantaneous mass interaction law because it has been revealed experimentally that the speed of propagation of a central Newtonian gravitational attraction is at least $2 \times 10^{10}\, c$, (Van Flandern 1998) where c is the speed of light. Such a velocity is experimentally indistinguishable from an instantaneous interaction. Consider the diagram of Figure 3 in which a particle of mass, m_0, in the laboratory is being acted on by an upward external force, $\vec{F}_e$. If the particle is free to move, it will accelerate with respect to the fixed stars (Machian inertial system) in the direction of $\vec{F}_e$ $(\theta = 0)$, perpendicular to the plane EE. If the inertial force, $\vec{F}_i$, is proportional to the magnitude of the acceleration, $\vec{a}$, and acts in the opposite direction, then it will increase from zero as soon as the particle begins to accelerate. The inertial force increases as the acceleration increases, ensuring that the force of inertia is always equal and opposite to the applied external force. This

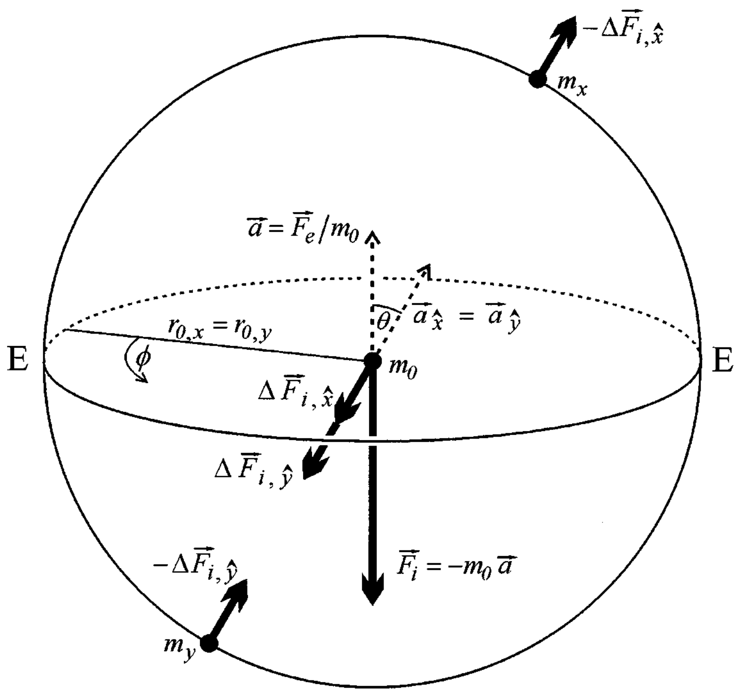

Figure 3. Machian inertial force interactions between an observable particle, m_0 and particles in the distant isotropic mass distribution (IDM), (m_x & m_y).

dynamic equilibrium is stable and thus determines the value of the acceleration that is caused by the application of a given external force. If the particle were to accelerate faster, then $\vec{F}_i$ would increase and retard the extra acceleration, Similarly, if the particle were to slightly decelerate, then $\vec{F}_i$ would decrease causing the particle to feel a net increased force in the $\vec{F}_e$ direction, thus resisting the deceleration. This stability caused by a real force is the mechanism behind Newton's 1st law, ensuring that an object does not accelerate with respect to the distant universe unless acted upon by another body.

If m_x is the mass of another particle as shown in Figure 3, then a repulsive Machian inertial interaction force, $\Delta F_{i,(0,x)}$, will act between m_0 and m_x which opposes their relative acceleration, $a_{0,x}$. The magnitude of the relative acceleration is quantified by

$$a_{0,x} = \frac{d^2 r_{0,x}}{dt^2}, \tag{13}$$

where $r_{0,x}$ is the distance between m_0 and m_x. We propose that the elemental inertial force law is a force of mass interaction which takes the form

$$\Delta F_{i(0,x)} = -K\, a_{(0,x)} \frac{m_0\, m_x}{r_{0,x}^n}, \tag{14}$$

where K is a dimensional constant whose meaning will be discussed later. Eq. (14) represents a mutual Newtonian force of attraction or repulsion between the two particles. It is positive, representing repulsion when $a_{0,x}$ is negative as a result of the two particles accelerating toward each other. Similarly, the force is negative, representing attraction when the two particles accelerate away from each other. In spite of the mathematical similarity with Newton's law of gravitation, Eq. (1), the Machian particle interaction, Eq. (14) is an additional force which vanishes when the two particles are not accelerating with respect to each other, even though they are still subject to mutual gravitational attraction.

A general expression for $\vec{a}_{\hat{x}}$, the acceleration vector of m_0 in the direction of m_x, defined by the unit vector, $\hat{x}$, can be formulated from the total acceleration vector, $\vec{a}$, and θ, the angle between the two as shown in Figure 3.

$$\vec{a}_{\hat{x}} = |\vec{2a}|\, \cos\theta\, \hat{x}. \tag{15}$$

Resolving in the $\hat{x}$ direction, Eq. (14) can define the inertial force on m_0 due to relative acceleration with respect to m_x as

$$\Delta\vec{F}_{i,\hat{x}} = -K\vec{a}_{\hat{x}} \frac{m_0\, m_x}{r_{0,x}^n} = -K|\vec{a}|\, \cos\theta \frac{m_0\, m_x}{r_{0,x}^n}\hat{x}. \tag{16}$$

Due to the Newtonian nature of the force described in Eq. (14), the reaction force on the particle, m_x, naturally has the same magnitude but the opposite direction as depicted in Figure 3.

If there is a particle, m_y, of the same mass as m_x and symmetrically opposed to m_x about m_0, then the interaction between m_0 and m_y can also be calculated. Since the distance $r_{0,y}$ increases as a result of the acceleration, $a_{\hat{y}}$, there is an attractive inertia force $\Delta F_{i,\hat{y}}$ of the same form as Eq. (16) which will also oppose the acceleration toward m_x. The two linear inertia forces on m_0 due to m_x and m_y therefore add together as shown in Figure 3, so that for the system of three masses

$$\vec{F}_{i,(x+y)} = \Delta\vec{F}_{i,\hat{x}} + \Delta\vec{F}_{i,\hat{y}} = 2\Delta\vec{F}_{i,\hat{x}}. \tag{17}$$

This leads to the important conclusion that using the force law proposed in Eq. (14), an isotropic mass distribution will lead to a non-zero inertia force on an accelerating particle.

It can thus be seen that the linear force of inertia between m_0 and any mass, m_x, will result in a downward directed component of $\Delta\vec{F}_{i,\hat{x}}$, perpendicular to EE and opposing $\vec{a}$. Using Eq. (16), it follows that the sum of the components resolved in the direction of the acceleration, $\hat{a}$, is

$$\vec{F}_{i,\hat{a}} = \sum_x \left|\Delta \vec{F}_{i,\hat{x}}\right| \cos\theta\, \hat{a} = -m_0\, \vec{a} \left(K \sum_x \frac{m_x \cos^2\theta}{r_{0,x}^n} \right). \tag{18}$$

The summation is taken over all of the particles in the universe. It will be seen that for an isotropic mass distribution, the inertial force components in the EE plane will cancel by symmetry.

We must now attempt to discover the power of the denominator, n, and provide an interpretation of the constant, K. To achieve this, it is helpful to rewrite Eq. (18) in spherical coordinates centred on m_0. It will be shown in the later discussion that the mass density of the observable universe is a function of the distance, r, from any observer. Therefore we can use the substitution,

$$\sum_x \frac{m_x}{r_{0,x}^n} = \int_0^{2\pi}\int_0^{\pi}\int_t^{\infty} \frac{\rho(r,\theta,\phi)\, r^2\, dr\, d\theta\, d\phi}{r^n}. \tag{19}$$

t is a non-zero distance which ensures there is no force singularity due to self-interaction. Its physical meaning will be discussed later. Eq. (18) can now be rewritten as

$$\vec{F}_{i,\hat{a}} = -m_0\, \vec{a}\, K \int_0^{2\pi}\int_0^{\pi}\int_t^{\infty} \rho(r,\theta,\phi)\, r^{2-n} \cos^2\theta\, dr\, d\theta\, d\phi. \tag{20}$$

From experience, we know that the magnitude of the force of inertia is independent of the direction of the observed acceleration, $\hat{a}$. In order for Eq. (20) to satisfy this condition, $\rho(r,\theta,\phi)$ must be invariant for all directions (θ,ϕ), and thus approximate to an isotropic density function which is purely dependent on distance, $\rho_i(r)$. To achieve this, we can write

$$\vec{F}_{i,\hat{a}} = -m_0\, \vec{a}\, K \left\{ \begin{array}{l} \left(\int_0^{2\pi}\int_0^{\pi}\int_t^{\infty} \rho_i(r)\, r^{2-n} \cos^2\theta\, dr\, d\theta\, d\phi \right) + \\ \left(\int_0^{2\pi}\int_0^{\pi}\int_t^{\infty} \rho_a(r,\theta,\phi)\, r^{2-n} \cos^2\theta\, dr\, d\theta\, d\phi \right) \end{array} \right\}, \tag{21}$$

where $\rho_a(r,\theta,\phi)$ describes the anisotropic density distribution defined by

$$\rho_a(r,\theta,\phi) = \rho(r,\theta,\phi) - \rho_i(r). \tag{22}$$

The direction invariance of $\vec{F}_{i,\hat{a}}$ implies that the first integral in Eq. (21) represents the dominating contribution to the inertial force from a very large isotropic mass distribution (IMD), while the second integral describes a negligible contribution to the inertial force as a result of interaction with a much smaller anisotropic mass distribution (AMD).

The AMD however is well known to us for it causes the gravitational forces that directly affect us, for instance those caused by the sun and moon and to a lesser extent the other bodies in the solar system. We know that our galaxy has a planar structure and thus must also be included in the (AMD). If n, the value of the power of $r_{0,x}^n$ in Eqs. (14)–(21), is taken to be 2, then using the anisotropic density distribution defined in Eq. (22), Newton's universal law of gravitation, Eq. (1), can be employed to describe the net gravitational force on m_0 in an arbitrary direction, $\hat{z}$, as

$$\vec{F}_{g,\hat{z}} = -G\, m_0\, \hat{z} \int_0^{2\pi} \int_0^{\pi} \int_t^{\infty} \rho_a(r, \theta, \phi)\, \cos\theta\, dr\, d\theta\, d\phi, \tag{23}$$

where G is Newton's gravitational constant and θ is the angle between dr and $\hat{z}$. Eq. (23) is valid because the contributions to the gravity force from the much larger isotropic mass distribution will come to zero by symmetry.

In the case of Eq. (23) and the second integral of Eq. (21), the value of t must be taken as any distance outside the test body, m_0, but less than the nearest interacting body. In the case of the first integral in Eq. (21), t must take on a value which represents the distance at which the anisotropic distribution, $\rho_a(r, \theta, \phi)$, becomes insignificant in the determination of the local value of $\rho(r)$. Observation indicates that such a distance is much larger than our galaxy or in fact much larger than our local cluster of galaxies. By inspection of recent maps of galaxies in the known universe, our best estimate of the distance at which this distribution becomes fairly isotropic, is in the region of $t = 70$–100 Mpc ($\sim 3 \times 10^8$ light years).

The dominance of the first integral in Eq. (21), as a result of inertial isotropy, allows us to neglect the second integral when we perform the integration, leaving

$$\vec{F}_{i,\hat{a}} = -m_0\, \vec{a} \left(\pi^2 K \int_t^{\infty} \rho_i(r)\, r^{2-n}\, dr \right). \tag{24}$$

In order to ensure that Eq. (24) remains equivalent to Eq. (5), Newton's empirical principle of inertia, we must ensure that the quantity in brackets is dimensionless and equal to unity. The integral in Eq. (24) can be represented by a constant whose value depends on the value of n. This implies that if

$$B_n = \int_t^{\infty} \rho_i(r)\, r^{2-n}\, dr \tag{25}$$

then

$$K = \frac{1}{\pi^2 B_n}, \tag{26}$$

and therefore Eq. (14), the Machian particle interaction that predicts the force of inertia can be rewritten as

$$\Delta F_{i(0,x)} = -\frac{1}{\pi^2 B_n} \frac{d^2 r_{0,x}}{dt^2} \frac{m_{0\,m_x}}{r_{0,x}^n}. \tag{27}$$

4. THE PARADOX OF A NEWTONIAN HOMOGENEOUS UNIVERSE

The major problem faced by this analysis so far is the possibility that B_n is infinite, since Eq. (25) represents an integration to infinity of the mass distribution in a possibly infinite universe. Since the time of Galileo, we have been aware that we are not occupying a privileged position in the universe. Consequently, until recently, it has been assumed that the universe is a fairly homogeneous distribution of matter with a constant density. Newton was aware that he was caught between two awkward universe scenarios, namely a) the apparently atheistic viewpoint that the universe was infinite in extent or b) that it represented a finite amount of matter in an infinite amount of space. The first, (a) was unsatisfactory from a theological and mathematical point of view and the second, (b) would imply that the universe should have collapsed as a consequence of his own law of universal gravitation. The debate regarding the validity of these two systems has developed further in the intervening 300 years (Jaki 1990, chap. 8) and ultimately led to one of Einstein's conjectures regarding a finite and curved space that led to the formation of his theory of General Relativity. The issue has usually been debated under the banner of the *Gravitational Paradox* and will now be investigated with regard to the proposed Machian inertial mass interaction force.

In an infinite homogeneous universe in which gravitational matter interactions are governed by the Newtonian inverse square law, $\vec{F}_{g,\hat{r}}$ is not in general a defined value. This can be demonstrated by dividing such a universe into two regions as shown in Figure 4. The surface of division is a spherical surface of radius, R, whose centre is at P. The test particle, m_0, lies on this surface. Newton (1962, Book I, Section XII, Prop. LXX, Theorem XXX) showed that a constant density spherical shell causes no net gravitational force on any particle inside the shell as a consequence of the inverse square force law. Consequently, there is no net gravitational force on m_0 due to matter outside the spherical dividing surface since it is surrounded by concentric spherical shells of constant mass density. Still assuming a homogeneous mass distribution, the gravitational force on m_0 due to the mass inside the surface can be calculated by assuming that the entire mass of

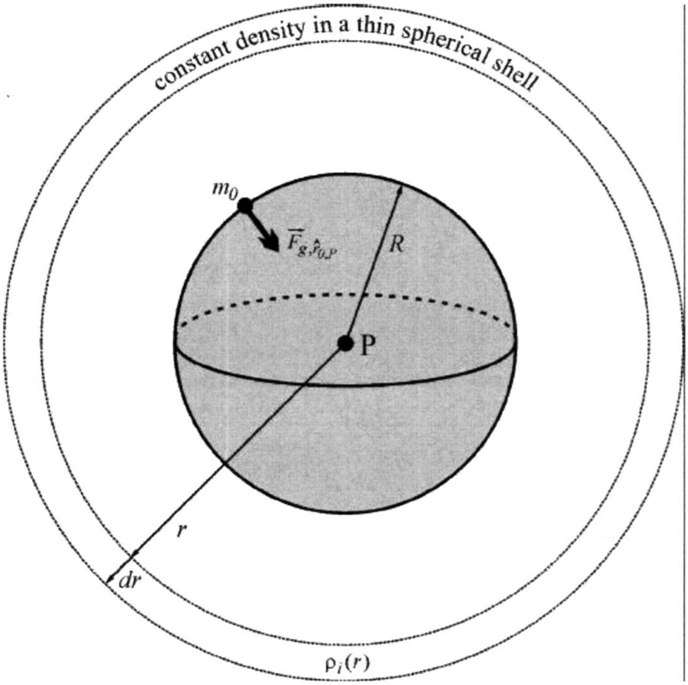

Figure 4. Demonstraction of the gravitational force due to an isotropic density function, $\rho_i(r)$, as described by Eq. (31). The universe is divided into two regions, (shaded and unshaded) centred on P, and containing concentric shells of constant mass density.

the shaded sphere is acting at the centre of mass, P, and thus the total force on m_0 is

$$\vec{F}_{g,\hat{r}_{0,P}} = G\frac{m_0\left(4/3\,\pi\ R^3\ \rho_h\right)}{R^2}\,\hat{r}_{0,P} = 4/3\,\pi\ G\,m_0\,\rho_h\,R\,\hat{r}_{0,P}, \tag{28}$$

where ρ_h is the density of the homogeneous mass distribution. Eq. (28) reveals that the magnitude and direction of the gravitational force is dependent on the arbitrary choice of the position of P which determines R. This demonstrates that the force of gravity as predicted by Newton's law of universal gravitation is undefined in an infinite homogeneous mass distribution.

5. THE DISCOVERY OF COSMIC HIERARCHICAL STRUCTURE

Recent astronomical measurements have indicated that contrary to the assumptions of the previous 300 years, the universe is not homogeneous, but appears to have a hierarchical structure, meaning that galaxy clusters are highly irregular and yet self-similar, with a fractal structure which is asymptotically dominated by voids. This isotropic structure has been described by Mandelbrot (1983, p. 87) as fractal homogeneity. There is general agreement that galactic structures are fractal up to a distance scale of 30–50 Mpc. Some have claimed that the data has revealed fractal correlations with dimension ($D \simeq 2$) up to the deepest scales probed to date (1000 Mpc) (Sylos Labini, Montuori & Pietronero 1998). These galaxies represent the furthest matter for which we have redshift data. The fractal dimension, D, is defined by

$$M \propto r^D, \tag{29}$$

where M is the mass of the matter contained in a sphere of radius, r, centred on any observer. Clearly, in a homogeneous distribution, ($D = 3$). There is a lively controversy regarding whether the mass distribution becomes homogenous at the largest length scales, which is an important feature of the Friedmann-Robertson-Walker (FRW) metric and the standard big-bang model of cosmology (Wu, Lahav & Rees 1999). Measured values of D were initially reported as low as ($D \sim 1.3$) (de Vaucouleurs 1970), but as more redshift measurements have become available, it has become clear that out to a depth of $\sim$50 Mpc, the galaxies appear to have a fractal distribution of dimension of ($D = 2 \pm 0.2$) (Roscoe 2002). It would appear that the analysis of the redshift data from the more distant galaxies is shrouded in controversy over the statistical methods used to analyse the data. However, there appears to be no observational basis preventing the conjecture that the entire universe has a fractal dimension of ($D \simeq 2$).

Using Eq. (29), we can define a fractal mass density, Z, to describe the mass contained in an arbitrarily positioned sphere of radius, R, such as the shaded one in Figure 4, as

$$Z = \frac{M_{sphere}(R)}{R^D}. \tag{30}$$

This implies that the mass density is constant in any given spherical shell, $(r + dr)$, but for ($D < 3$) the density of each shell decreases as r increases (Mandelbrot 1983, p. 88). Using Eq. (30), an isotropic density function in a fractal distribution of dimension, D, can be defined as

$$\rho_i(r) = \frac{D\,Z}{4\pi} r^{D-3}. \tag{31}$$

We can now calculate the Newtonian gravitational force on the particle, m_0, in the arbitrary direction $\hat{r}_{0,P}$ in an isotropic fractal universe again using Figure 4. The mass outside the sphere is still in spherical shells of constant density and thus causes no net gravitational force on m_0. Even in a fractal distribution, the centre of mass of the sphere remains at its centre, P. However, the total mass contained in the shaded sphere can now be expressed from Eq. (30) as

$$M_{sphere} = Z\, R^D. \tag{32}$$

Therefore, in analogy with Eq. (28), the gravitational force on m_0, in the arbitrary direction $\hat{r}_{0,P}$, in a fractal mass distribution can be written

$$\vec{F}_{g,\hat{r}_{0,P}} = G\,\frac{m_0\, M_{sphere}}{R^2}\hat{r}_{0,P} = G\frac{m_0\, Z\, R^D}{R^2}\hat{r}_{0,P}. \tag{33}$$

It can be seen from Eq. (33) that if ($D \leq 2$), then the ill defined force of gravity in any arbitrary direction and acting on every particle, m_0, due to an isotropic universe goes to zero in an infinite isotropic distribution. In this situation, the well defined direction and magnitude of the observable gravitational force is completely caused by the AMD as described by Eq. (23). The observation that ($D \simeq 2$) is consistent with the requirement that ($D \leq 2$) in order to ensure the resolution of the *Gravitational Paradox*. Hoyle (1953) and Mandelbrot (1983, p. 92) have both speculated on this intimate connection between the hierarchical structure of the cosmos and Newtonian gravitation, and have suggested that it may be the force of Newtonian gravity that creates the fractal structure that we observe.

In our treatment of Mach's principle we measure local accelerations against a background of point like galaxies, apparently fixed relative to each other over the short timescales of human experience. This is similar to Roscoe (2002) who has developed a model universe consisting, initially, of a stationary (but not static) ensemble of identical particles existing in a formless continuum, without preconceived notions of clocks and measuring rods. He concludes that, on very large scales, all motion can be considered to be inertial, and the distribution of mass is necessarily fractal with dimension ($D = 2$).

Einstein and his colleagues were apparently unaware that a fractal mass distribution such as described by Eqs. (29)–(32) was possible, in a manner that does not pre-select any unique position as the centre of the universe. Consequently, he felt forced to propose a geometry of curved space in which the mass contained in the universe was finite (to avoid the gravitational paradox) but yet the universe was unbounded, in that space was curved so that all of space could be filled with a homogeneous but finite matter distribution (Einstein 1920, p. 108). While the mathematics behind fractal geometry was slowly emerging in the late 19th century, its application to the study of nature was only first attempted in the 1970's by Mandelbrot (1983). Thus Einstein was unaware of the power of such a matter distribution for the purpose of resolving the gravitational paradox. With the even more recent experimental confirmation of a fractal ($D \simeq 2$) mass distribution of the galaxies up to the limits of our measuring equipment, there seems to be no longer a conceptual requirement to abandon Newtonian dynamics or Euclidian space.

The discovery that D is less than or equal to 2 may resolve the gravitational force paradox for an infinite universe under the influence of Newtonian gravitation, but it is not sufficient to prevent infinite gravitational potentials. However, since we never directly measure potentials, but only accelerations which are proportional to forces, then infinite potentials with finite gradients are not a physical problem.

It is often claimed that the universe consists of up to 99% unobservable dark matter. By definition, we know nothing directly about the nature or distribution of this material. Dark matter distributions are conventionally invoked when a gravitational theory is unable to explain the behaviour of observable bright matter. For the purposes of this paper, the observed ($D \simeq 2$) distribution of bright matter and its natural relationship with Newtonian gravity, offers no reason to suspect that if there is dark matter that it should be distributed differently.

Returning to the force of inertia, in order to ensure that it is always finite, we must confirm that B_n as defined by Eq. (25) remains finite. Using the relationship for isotropic density, $\rho_i(r)$, in a fractal distribution given in Eq. (31), we can write

$$B_n = \frac{D\,Z}{4\pi} \int_t^{\infty} r^{D-n-1} dr. \tag{34}$$

For B_n to remain finite, $n > D$.

In the same manner that Hoyle (1953) and Mandelbrot (1983, p. 92) claimed a connection between the inverse square law of gravitation and the hierarchical ($D \simeq 2$) structure of the universe, we propose that this fractal mass distribution also implies that a mass interaction law of inertia will also be an inverse square interaction ($n = 2$). If this were so, then ($n = D = 2$) is the limiting case and in order to maintain a finite value of inertia, D must actually be less than 2. It is plausible that the universe is constantly trying to achieve a homogeneous distribution ($D = 3$), but that as it approaches $D \simeq 2$ it cannot get beyond there because at that point, the inertial force would become infinite and all motion would cease. Since $D \simeq 2$ is the observed universal fractal dimension, we feel justified to assume that the inertial force is an inverse square law. This assumption seems quite natural when it is noted that all the Newtonian matter interaction force laws discovered to date are built on the inverse square relation.

Several important pieces of information can now be assimilated in order to arrive at a plausible expression for the proposed Machian inertial matter interaction law.

1) The exponent of the distance of separation in Newton's law of universal gravitation has been proved to very high accuracies even down to length scales of 200 μm. No deviation from the inverse square law of gravitational matter interaction has been detected (Hoyle et al. 2001).

2) The most recent surveys of the cosmos imply that matter on a large enough scale is distributed in an isotropic, but inhomogeneous manner with a fractal dimension, $D \simeq 2$.
3) The force of gravity is measurable and well defined on all objects. The resolution of the *Gravitational Paradox* by using a fractal mass distribution implies that $D \leq 2$, which is consistent with the previous two points. In order for our proposed Machian inertial matter interaction law, Eq. (27), to predict a finite force of inertia, it is essential that $D < n$ and therefore $n \geq 2$.
4) The measured force of inertia is proportional to acceleration and acts to oppose an external force applied to an object. It always has finite magnitude.

Point (4) highlights the finite nature of the inertial force and in order to ensure this behaviour and also to absorb the suspected connection between the universal fractal dimension and the proposed inertial force law, points (1–3) justify our use of $n = 2$. Consequently, there are now two good reasons (well defined gravity forces and finite inertial forces) to suspect that the observed hierarchical structure of the universe is a consequence of our proposed Machian inertial force law which is closely related to the Newtonian gravitational law.

6. CONCLUSIONS

Eq. (27), the elemental form of the proposed Machian inertial matter interaction law, can now be justified as containing $n = 2$, and can then be summed over all particles in the universe yielding the total inertial force on a particle, m_0 as

$$\vec{F}_{i,\hat{a}} = -\frac{1}{\pi^2 B} m_0 \vec{a} \sum_x \left(\frac{m_x}{r_{0,x}^2} \cos^2 \theta \right). \tag{35}$$

It is also claimed that Eq. (27) is a finite instantaneous action at a distance force between particles of matter which creates a resistance to the acceleration caused by an external force acting on one of them. Our knowledge of the fractal distribution of matter throughout the universe combined with the finite, inverse square nature of the gravitational force allow us propose that $n = 2$ for both the gravitational and inertial force laws and D is approximately equal to but slightly less than 2. Therefore the elemental form of our proposed inertial force law, Eq. (27), can be expressed as

$$\Delta F_{i(0,x)} = -\frac{1}{\pi^2 B} \frac{d^2 r_{0,x}}{dt^2} \frac{m_{0\, m_x}}{r_0, x^2} = \frac{1}{\pi^2 B G} a_{(0,x)} \Delta F_{g(0,x)}. \tag{36}$$

Eq. (36) therefore represents an instantaneous Newtonian force of either attraction or repulsion between mass particles that is proportional to their relative acceleration, $a_{(0,x)}$, and is also proportional to the gravitational force between the objects, $\Delta F_{g(0,x)}$. It is also inversely proportional to a constant B described by

$$B = \frac{D\,Z}{4\pi} \int_{t}^{\infty} r^{D-3}\,dr. \tag{37}$$

where Z is a universal fractal mass density defined by Eq. (30), D is the dimension of the fractal distribution and t is the radius from m_0 of a spherical shell in which the mass density distribution becomes dominantly isotropic. If $D < 2$, then B must have a finite value, but we need a much more precise measure of D, Z and t in order to put a magnitude to it.

Eq. (36) invites a brief speculation regarding the very precisely measured, but nevertheless mysterious, Newtonian gravitational constant, G (6.67×10^{-11} m^3 kg s^{-2}). If the infinite cosmos was expanding in such a way that every object was accelerating from every other with an acceleration of ($\pi^2 B\,G$), then our proposed force of inertia would become the cause of the gravitational force. With ever increasing cosmological observations, it will eventually be within our powers to estimate B (kg m^{-2}) in Eq. (37), and thus our local laboratory determination of G may be the measurement of a universal expansion acceleration. This unexpected acceleration may be the mechanism by which the universe avoids becoming homogeneous and retains its hierarchical structure. However this pure speculation is only built on the rather hopeful human desire to unify the known force laws and cannot be justified by any existing experimental knowledge.

More importantly, Eq. (36) complements Newton's universal law of gravitation and thereby completes Newton's theory of instantaneous action at a distance mechanics in a manner which answers the cosmological doubts of both Mach (absolute space) and Einstein (gravitational paradox) which were responsible for the general relativistic revolution. Recent knowledge of the hierarchical structure of the universe and the consequent finite nature of our proposed inertial force law opens the door for a return to a simpler cosmological model, based on Newtonian forces between pieces of matter, acting in a Euclidean geometry. It is important to remember that Newtonian forces and Euclidean geometry have never been found in error in any laboratory controlled experiment and are still used with complete accuracy to predict the motion of all man-made objects in our solar system. A famous apparent discrepancy is the anomalous precession of the perihelion of mercury, but it represents an example of an uncontrolled experiment in which the variables such as solar oblateness and mass distribution cannot be independently manipulated and thus it lacks the rigor with which Newtonian theory has been evaluated.

To summarise our proposed interaction mechanism, the Newtonian elemental inertial force, Eq. (36), always acts as an attraction or repulsion between the two bodies, m_0 and m_x, at the same time as an external applied force acts on one of them, m_0. The inertial force always opposes the relative acceleration between m_0 and every body, m_x. In the spirit of Mach's principle, summing over all objects in the universe yields a finite value for the force of inertia on an accelerating particle. Employing the now well confirmed fractal matter distribution consistent with ($D < 2$), the finite magnitude of the force of inertia occurs despite the infinite number of non-cancelling instantaneous interactions. The mass related force of inertia is therefore responsible for controlling the magnitude of the accelerations that are caused by applied forces and is the mechanism that lies behind Newton's 2nd law of motion.

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