Construction of Efficient Mixed-Level Fractional Factorial Designs

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Mixed-level factorial experimental designs involve factors with different numbers of levels. Full factorial designs require runs at all possible combinations of the factor levels. As the number of factors and/or factor levels increases, the total number of experiments increases dramatically. As a result, interest has focused on developing orthogonal or near-orthogonal mixed-level fractional factorial designs. Currently existing mixed-level designs are all balanced. However, relaxing the requirement of balance may result in a reduced number of experimental runs in practice. The objective of this paper is to develop mixed-level fractional factorial designs with economical run sizes that are as nearly balanced and orthogonal as possible. A new criterion is developed to assess the degree of near-balance for comparing and constructing designs. A modified J_2-optimality criterion is used for evaluating design near orthogonality. These criteria are combined and used to assess different design alternatives. Three algorithms are then compared and used to build designs with desirable combinations of near balance and near orthogonality.

Key Words: Balance; Columnwise-Pairwise Algorithm; Coordinate Exchange Algorithm; Design of Experiments; Desirable Properties; Genetic Algorithm; Orthogonality.

TRADITIONAL two-level factorial designs are widely used in industrial research and development. In some situations, however, factors with more than two levels are required, especially when those factors have qualitative levels. Accordingly, mixed-level factorial designs are employed. Two desirable properties used in assessing the adequacy of mixed-level designs are balance and orthogonality. Balance requires that each level of a factor be run the same number of times in an experiment, resulting in an even distribution of information for each factor level. Orthogonal designs are column pairwise linearly independent and are useful in assessing factor significance. Constructing mixed-level designs with balance, small run sizes, and conditions approaching orthogonality has been a focus in the literature. Several authors have developed algorithms to build balanced orthogonal mixed-level designs. Additionally, balanced near-orthogonal designs have been generated as alternatives to strictly orthogonal designs when orthogonal designs are either difficult or impossible to produce.

Wang and Wu (1991) first proposed an approach for constructing orthogonal mixed-level designs based on difference matrices. This construction method applies the generalized Kronecker sum and uses the technique of adding columns. Wang (1996) introduced another distinct method, also based on difference matrices that can produce a new class of designs. DeCock and Stufken (2000) proposed an algorithm for constructing orthogonal mixed-level designs using existing two-level orthogonal designs. Wang and Wu (1992) and Nguyen (1996) constructed near-orthogonal mixed-level designs. Xu (2002) proposed an algorithm to construct orthogonal and near-orthogonal designs based on the concept of J_2-optimality. The J_2-optimality is equivalent to several other orthogonality criteria, including the (M, S) criterion (Eccleston and Hedayat (1974)), the A_2-optimality criterion (Xu (2002)), the $B(2)$ criterion

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(Lu et al. (2006)), and the ave(s^2) criterion (Booth and Cox (1962)). However, the J_2-optimality is more appropriate for situations where factor levels are not denoted by contrasts because J_2-optimality does not require computing the value of $|X^{\mathrm{T}}X|$, where X is the contrast matrix of the design. The theory of orthogonal designs was systematically discussed by Hedayat et al. (1999).

The number of runs required for factorial designs increases dramatically as the number of factors or factor levels increases. Fractionating mixed level factorials often necessitates that balance be compromised. For example, consider a design with three factors: one with 3 levels, one with 4 levels, and a third with 5 levels. The full factorial design, represented as $3^1 4^1 5^1$, involves 60 runs and balance is maintained. Any fraction of the 60 runs will sacrifice balance. Suppose an engineer only has resources for 30 tests and the test objective is factor screening. The intent of this paper is to introduce efficient mixed-level fractional factorial designs that are as nearly balanced and orthogonal as possible and capable of meeting scarce-resource requirements. Efficient mixed-level fractional factorial designs are emphasized, conserving resources (runs) while obtaining the best possible balance and orthogonailty properties. The design construction process involves evaluating balance and orthogonality properties by proposing, studying, and implementing relevant criteria. By using these design balance and orthogonality criteria in combination with design-selection algorithms, efficient near-balanced, near-orthogonal mixed-level designs can be constructed.

A new optimality criterion, the balance coefficient, will be defined and formulated. J_2-optimality is a useful measure of design orthogonality, although it is not invariant to design size. As such, a modified J_2-optimality is proposed for measuring the degree of orthogonality across designs with different run sizes. Different types of algorithms are compared to construct efficient mixed-level designs.

In the next section, we propose and formulate a balance coefficient and modify the J_2-optimality criterion. Using these balance and orthogonality criteria, three algorithms are then compared for constructing efficient mixed-level designs. The performance of these algorithms is discussed in the subsequent section. Finally, examples are provided of generating near-balanced near-orthogonal mixed-level fractional factorial designs.

Optimality Criteria for Mixed-Level Designs

Factor levels are coded to measure the balance and orthogonality properties of mixed-level designs. A new criterion, the balance coefficient, is developed to measure the degree of balance for mixed-level designs. A standardized J_2-optimality criterion is used in assessing design near-orthogonality. These criteria are combined to generate designs containing both desirable properties.

Balance Coefficient for Factorial Designs

In a balanced design, each factor level occurs the same number of times, so there is consistency in the variance of the difference of observations at two treatment combinations. With mixed-level designs, the "1, 2, 3,..." representation is commonly used for factor levels. For an $n \times m$ design matrix D, let n represent the number of rows and m the number of columns. Rows correspond to runs and columns to factors. Column j contains l_j levels and c_{rj} is the number of times the rth level appears in that column. Let $\mathbf{c}_j = [c_{1j}, c_{2j}, \ldots, c_{l_j j}]^{\mathrm{T}}$ be the counts for each level for column j.

For example, consider a half fractional $2^1 3^1 4^1$ design D with 12 runs (Figure 1). The corresponding values of parameters for D are shown in Figure 2.

For a factor j, the degree of unbalance can be determined by the $\mathbf{c}_j$. When the c_{rj} for all r equal a common value, then factor j is balanced. Because $\sum_{r=1}^{l_j} c_{rj} = n$, $\mathbf{c}_j$ can be represented by a $l_j - 1$ dimensional coordinate hyperplane. For the design D in Figure 1, the number of levels for the second factor can be expressed as $c_{12}+c_{22}+c_{32} = 12$. This equation has two degrees of freedom and can be represented

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 2 & 4 \\ 2 & 3 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

FIGURE 1. A Half Fractional Factorial $2^1 3^1 4^1$ Design.

Parameter	Value
n	12
m	3
$l = [l_1, l_2, l_3]$	$[2, 3, 4]$
$\mathbf{c}_1 = [c_{11}, c_{21}]$	$[6, 6]$
$\mathbf{c}_2 = [c_{12}, c_{22}, c_{32}]$	$[4, 4, 4]$
$\mathbf{c}_3 = [c_{13}, c_{23}, c_{33}, c_{43}]$	$[3, 3, 3, 3]$

FIGURE 2. Parameters for $2^1 3^1 4^1$, 12 Run Design.

by a two-dimensional coordinate plane in a three-dimensional coordinate system (Figure 3).

All the feasible solutions of $c_{12}+c_{22}+c_{32} = 12$ are scattered on this plane. Figure 4 shows some of the feasible solutions. The remaining feasible points are not shown because they are symmetric with regard to the center point. The point in the center, $[4, 4, 4]$, represents a balanced column. In general, a column with l_j levels can be characterized by an equation, $\sum_{r=1}^{l_j} c_{rj} = n$, which denotes a $l_j - 1$ dimensional hyperplane. We employ a distance function and define

$$H_j = \sum_{r=1}^{l_j} (c_{rj} - T)^2$$

as the column j balance coefficient, where the center point of the hyperplane $T = n/l_j$ is fixed. Using this definition, H_j becomes

$$H_j = \sum_{r=1}^{l_j} \left(c_{rj} - \frac{n}{l_j} \right)^2 .$$

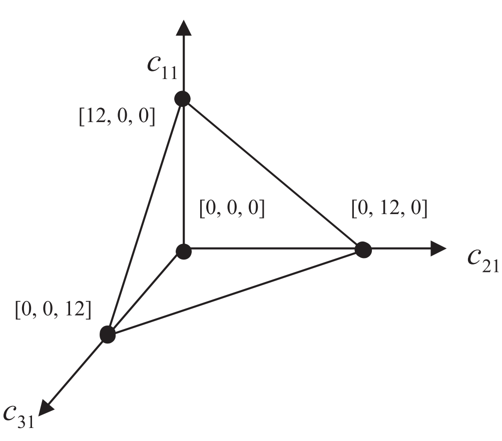

FIGURE 3. Graphical Representation of the Concept of Distance Associated with the Balance Coefficient for a 3-Level, 12-Run Column.

The balance coefficient H for the design matrix is defined as

$$H = \sum_{j=1}^{m} H_j = \sum_{j=1}^{m} \sum_{r=1}^{l_j} \left(c_{rj} - \frac{n}{l_j} \right)^2 .$$

The balance coefficient H is a function of n and it can be standardized to be independent of n. The standardized frequency for a specific level is defined as $f_{lj} = c_{rj}/n$. If f_{rj} is used to replace c_{rj}, then standardized H_j and H can be given by

$$\hat{H}_j = \sum_{r=1}^{l_j} \left(f_{rj} - \frac{1}{l_j} \right)^2$$

and

$$\hat{H} = \sum_{j=1}^{m} \hat{H}_j = \sum_{j=1}^{m} \sum_{r=1}^{l_j} \left(f_{rj} - \frac{1}{l_j} \right)^2 . \qquad (1)$$

When the design is balanced, $f_{rj} = (1/l_j)$ and $c_{rj} = (n/l_j)$ for all levels l_j in column j, and the distance function reaches its minimum value, which is $\hat{H}^* = 0$, where $\hat{H}^* = \min_{f_{rj} \in [0,1]}(\hat{H})$.

Standardized J_2-Optimality

Besides the property of balance, orthogonality is also a useful property of design quality that can be integrated into a combined design criterion. Consider an $n \times m$ matrix $D = [a_{ij}]$, where a_{ij} are the elements of D. The coincidence between two elements a_{ij} and a_{kj} is defined by $\delta(a_{ij}, a_{kj})$, where $\delta(a_{ij}, a_{kj}) = 1$ if $a_{ij} = a_{kj}$ and 0 otherwise. The value of $\sum_{j=1}^{m} \delta(a_{ij}, a_{kj})$ measures the coincidence between the ith and jth rows of D. The design D is orthogonal if the coincidence of every two rows of D is minimized (Xu (2002)). The J_2-optimality criterion is defined by Xu (2002) as

$$J_2(D) = \sum_{1 \le i \le j \le n} \left[\sum_{j=1}^{m} \delta(a_{ij}, a_{kj}) \right]^2 .$$

For a fixed number of runs, a design is J_2-optimal if it minimizes J_2. A balanced design D is orthogonal if it is J_2-optimal (Xu (2002)). J_2-optimality can be standardized to be independent of the design size in a way similar to the standardization of the balance criterion. One way to standardize J_2-optimality is to use the average coincidence. For any two rows, a_i and a_j, $\delta(a_i, a_k)$ is the number of coincidences between the ith and kth rows. The standardized coincidence

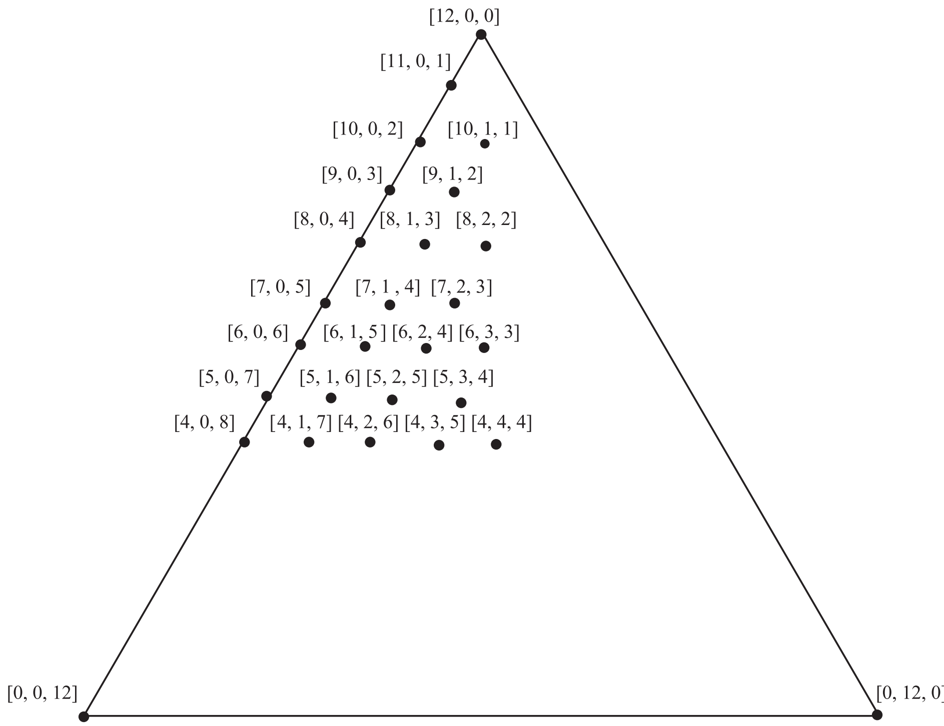

FIGURE 4. Partial Feasible Solutions of $c_{12} + c_{22} + c_{32} = 12$.

is defined as

$$\hat{\delta}(a_i, a_k) = \frac{\delta(a_i, a_k)}{m} = \frac{\sum_{j=1}^{m} \delta(a_{ij}, a_{kj})}{m},$$

where the number of factors m is the length of the row. The interpretation of this standardized coincidence value is the degree of similarity between two rows. Then the average of squared standardized coincidences is defined as

$$\text{ave}(\hat{\delta}(D)) = \frac{\sum_{i=1}^{n} \sum_{k=i+1}^{n} \left[\hat{\delta}(a_i, a_k)\right]^2}{\binom{n}{2}}.$$

For a fixed n, designs with smaller $\text{ave}(\hat{\delta}(D))$ values are more orthogonal. However, for increasing n, designs with larger $\text{ave}(\hat{\delta}(D))$ values are more orthogonal. That is, the optimal $\text{ave}(\hat{\delta}(D))$ is monotonically increasing with n. Therefore, a parameter, $1/n$, is used to adjust the $\text{ave}(\hat{\delta}(D))$ so that the standardized orthogonality measure will decrease as n increases. A standardized J_2-optimality is then proposed and defined as

$$\hat{J}_2(D) = \frac{1}{n} \frac{\sum_{i=1}^{n} \sum_{k=i+1}^{n} \left[\hat{\delta}(a_i, a_k)\right]^2}{\binom{n}{2}}. \qquad (2)$$

Figure 5 shows an example for a 2^{3-1} balanced orthogonal design. Applying the definitions, $J_2(D) = 6$ and $\hat{J}(D) = 0.0278$, which is J_2 optimal and orthogonal. Optimality is verified using the lower bound equation in Xu (2002).

Other Optimality Criteria

Other than the standardized balance coefficient and a standardized J_2-optimality, several other criteria have been proposed for evaluating mixed-level designs. Yamada and Lin (2002) constructed mixed-level supersaturated designs that maximize the value of a χ^2 statistic, which is used for a measure of dependency of the design columns. Additional discussion regarding χ^2 dependency is given by Yamada and Matsui (2002). Fang et al. (2003a) used a criterion called $E(f_{\text{NOD}})$ to measure nonorthogonality of mixed-level supersaturated designs. Fang et al. (2003b), Xu and Wu (2001), and Mukerjee and Wu

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

FIGURE 5. A $2^{3\text{-}1}$ Design.

(2001) discussed the construction of minimum aberration mixed-level designs.

Mixed-Level Design Construction Algorithms

One commonly used notation for characterizing orthogonal fractional factorial mixed-level designs is $OA(n, l_1^{k_1}, l_2^{k_2} \cdots l_T^{k_T})$, which denotes a column pairwise orthogonal array with number of runs n, having $\sum_{t=1}^{T} k_t$ columns (factors) such that there are k_t factors with l_t levels. The notation $EA(n, l_1^{k_1}, l_2^{k_2} \cdots l_T^{k_T})$ is introduced here to denote a near-balanced, near-orthogonal, efficient mixed-level fractional design.

With a large number of factors and/or factor levels, finding efficient optimal designs can be a large nonlinear integer programming problem. Approximation algorithms such as the genetic algorithm (GA) can be considered. For a detailed discussion of using the GA for constructing optimal designs, see Guo (2003), Borkowski (2003), Heredia-Langner et al. (2003), Heredia-Langner et al. (2004), and Ortiz et al. (2004).

Starting from randomly generated initial designs, a GA can be designed for constructing mixed-level designs using a combination of the standardized balance coefficient and the standardized J_2-optimality as a single objective function. We consider using $Z = a\hat{J}_2(D) + b\hat{H}(D)$ as an objective function, where a and b are nonnegative coefficients for adjusting the relative weight of the two criteria. Preliminary studies showed that such a GA is insensitive to values of a and b. If the designs generated by this GA are not near balanced (by observing $\mathbf{c}_j$), the designs can be further improved by compelling the factor levels to distribute more evenly. This additional procedure actually helps a GA generate efficient mixed-designs with optimal near-balance property and with improved near orthogonality. More details with respect to tuning parameters of this GA and the MATLAB source codes of this GA can be found in Guo (2003).

Besides the GA, exchange algorithms can also be used to construct efficient mixed-level designs. Nguyen and Miller (1992) reviewed several exchange algorithms (Fedorov (1972), Mitchell and Miller (1970), Wynn (1970), Mitchell (1974), and Atkinson and Donev (1989)). The two types of exchange algorithms are the row-exchange algorithm and the column-exchange algorithm. Meyer and Nachtsheim (1995) proposed a row-exchange algorithm called a coordinate exchange algorithm to construct exact optimal experimental designs. The coordinate exchange algorithms are used by many commercial softwares, such as JMP and DESIGN EXPERT. The coordinate exchange algorithm starts from a random initial design and does not guarantee balance at intermediate or final iterations. However, the additional procedure to compel the factor levels to distribute more evenly can also improve designs generated from coordinate exchange algorithm in terms of balance property as well as orthogonality property.

Unlike the GA and the row-exchange algorithm, column-exchange algorithms (Li and Wu (1997), Xu (2002)) maintain the best possible balance property at each iteration. The algorithm proposed by Xu (2002) is an example of a column-exchange algorithm. See Xu (2002) for more information on his algorithm. The C source code is available at Xu's website. The basic idea is to add balanced/near-balanced columns sequentially to existing columns based on the J_2-optimality criterion. However, the J_2-optimality was standardized in this paper so that design orthogonality can be compared for designs with a different number of runs.

The genetic, coordinate exchange, and column-wise-pairwise algorithms were each used to construct several efficient mixed-level designs for the purpose of comparison. The Appendix provides the best of these designs along with their factor-level counts $\mathbf{c}_j$. In general, optimal designs are featured with superior balance and superior orthogonality. In terms of algorithm performance (Table 1), the columnwise–pairwise approach consistently performs better than the genetic and coordinate exchange approaches, although it is not known whether the generated designs are indeed optimal. Both the genetic algorithm and the coordinate exchange algorithm perform well, but in general, the designs from these two algorithms are inferior to those from the columnwise–pairwise algorithm. Comparing the designs by individual criterion, designs constructed using both the genetic algorithm and the columnwise–pairwise algorithm had the optimal balance values and were consistently better than designs constructed by the coordinate exchange algorithm. In terms of orthogonality, the columnwise–pairwise algorithm performs the best, only slightly better than coordinate exchange and generally better than the genetic algorithm.

An Example

Consider a situation where there are three factors, one with three-levels, one with five-levels, and

TABLE 1. Comparison of *EA*s Constructed by Three Algorithms

	Genetic algorithm		Coordinate exchange		Columnwise–0airwise	
Efficient designs	$\hat{H}$	$\hat{J}_2$	$\hat{H}$	$\hat{J}_2$	$\hat{H}$	$\hat{J}_2$
$EA(20, 2^4 3^1 4^1)$	0.00028	0.009510	0.00280	0.009325	0.00028	0.009305
$EA(20, 2^4 3^1 5^1)$	0.00028	0.009205	0.00110	0.008845	0.00028	0.008840
$EA(15, 2^1 3^1 5^1 7^1)$	0.00150	0.005513	0.00150	0.005513	0.00150	0.005513
$EA(21, 2^6 3^1 5^1 7^1)$	0.00096	0.007610	0.00200	0.007414	0.00096	0.007381
$EA(30, 2^6 3^1 5^1 7^1)$	0.00018	0.005637	0.00140	0.005533	0.00018	0.005493
$EA(21, 3^2 5^1 7^1)$	0.00045	0.003924	0.00160	0.003343	0.00045	0.003329
$EA(21, 3^1 4^1 7^1)$	0.00057	0.004057	0.00057	0.003705	0.00057	0.003705
$EA(21, 3^2 4^1 7^1)$	0.00043	0.004648	0.00043	0.003657	0.00043	0.003657
$EA(20, 2^3 5^1 7^1)$	0.00043	0.007400	0.00043	0.006935	0.00043	0.006895
$EA(20, 2^4 5^1 7^1)$	0.00036	0.008280	0.00200	0.007480	0.00036	0.007435
$EA(20, 3^1 4^1 5^1)$	0.00056	0.005175	0.00390	0.004530	0.00056	0.004475
$EA(20, 2^3 3^1 4^1 5^1)$	0.00028	0.007480	0.00110	0.007075	0.00028	0.007040
$EA(20, 2^4 3^1 4^1 5^1)$	0.00024	0.007980	0.00170	0.007555	0.00024	0.007480
$EA(28, 2^3 6^1 7^1)$	0.00034	0.005343	0.00034	0.005093	0.00034	0.005093
$EA(28, 2^4 6^1 7^1)$	0.00028	0.005936	0.00071	0.005529	0.00028	0.005521
$EA(21, 3^1 6^1 7^1)$	0.00110	0.003452	0.00110	0.002948	0.00110	0.002948
$EA(24, 4^1 6^1 7^1)$	0.00099	0.002617	0.00099	0.002113	0.00099	0.002113
$EA(24, 5^1 6^1 7^1)$	0.00150	0.002317	0.00150	0.001879	0.00150	0.001879
$EA(24, 3^1 5^1 6^1 7^1)$	0.00110	0.002738	0.00200	0.002200	0.00110	0.002188
$EA(20, 4^1 5^1 6^1 7^1)$	0.00140	0.002385	0.00260	0.001940	0.00140	0.001860

one with seven-levels. The full $3^1 5^1 7^1$ factorial design contains 105 runs, and all 105 runs are required to make the design balanced, so no mixed-level fractional will have $\hat{H} = 0$. In this situation, efficient designs with near-balance and near-orthogonality properties are needed. Three designs will be constructed using the columnwise–pairwise algorithm: $EA(15, 3^1 5^1 7^1)$, $EA(21, 3^1 5^1 7^1)$ and $EA(30, 3^1 5^1 7^1)$. By using design sizes of 15, 21, and 30, at least two factors in each design can be balanced, making it fairly easy to assess balance performance and accordingly the algorithm performance. For each design, the standardized balance coefficient and standardized J_2-optimality values are given for comparison.

The first design generated is $EA(15, 3^1 5^1 7^1)$. Figure 6 gives this design and its factor level count vectors $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$. Both of the three-level and five-level factors are balanced, and although the seven-level factor is not balanced, the algorithm has allocated the levels such that the corresponding standardized distance from the centroid of the coordinate plane is minimized. The seven-level factor count is $\mathbf{c}_3 = [2, 3, 2, 2, 2, 2, 2]$. The standardized balance coefficient $\hat{H} = 0.0013$, which can be seen from $\mathbf{c}_3$, is the best value for this design. The standardized orthogonality value of 0.0038 was the best value obtained from 20 passes through each of the all three types of algorithms. No lower bound of standardized

$$D = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 2 & 2 \\ 1 & 3 & 5 \\ 1 & 4 & 4 \\ 1 & 5 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 5 \\ 2 & 3 & 6 \\ 2 & 4 & 7 \\ 2 & 5 & 1 \\ 3 & 1 & 4 \\ 3 & 2 & 1 \\ 3 & 3 & 2 \\ 3 & 4 & 3 \\ 3 & 5 & 6 \end{bmatrix} \qquad [\,\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3\,] = \begin{bmatrix} 5 & 3 & 2 \\ 5 & 3 & 3 \\ 5 & 3 & 2 \\ & 3 & 2 \\ & 3 & 2 \\ & & 2 \\ & & 2 \end{bmatrix}$$

FIGURE 6. $EA(15,\ 3^1 5^1 7^1)$ and Its Factor Level Count Vectors.

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 3 & 3 \\ 1 & 5 & 4 \\ 1 & 4 & 5 \\ 1 & 3 & 6 \\ 1 & 2 & 7 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 5 & 3 \\ 2 & 2 & 4 \\ 2 & 5 & 5 \\ 2 & 4 & 6 \\ 2 & 3 & 7 \\ 3 & 4 & 1 \\ 3 & 3 & 2 \\ 3 & 1 & 3 \\ 3 & 1 & 4 \\ 3 & 3 & 5 \\ 3 & 2 & 6 \\ 3 & 5 & 7 \end{bmatrix} \qquad [\,\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3\,] = \begin{bmatrix} 7 & 4 & 3 \\ 7 & 4 & 3 \\ 7 & 5 & 3 \\ & 4 & 3 \\ & 4 & 3 \\ & & 3 \\ & & 3 \end{bmatrix}$$

FIGURE 7. $EA(21,\ 3^1 5^1 7^1)$ and Its Factor Level Count Vectors.

J_2-optimality for near-balanced designs is currently available for comparison.

We now consider designs requiring 21 runs for the same factors, resulting in an $EA(21, 3^1 5^1 7^1)$ (Figure 7). The factor level count vectors of this design are also provided in Figure 7. In this case, both three- and seven-level factors are balanced, and the five-level factor is the most balanced. The balance coefficient of this design is 0.0006 and the standardized J_2-optimality is 0.0033.

A third case considers a design with 30 runs, EA(30, 315171). The generated design (Figure 8) is balanced in the three- and five-level factors, while the seven-level factor is the best possible balance. The balance coefficient of this design is 0.00053 and the standardized J_2-optimality is 0.0026. Again, by inspecting $\mathbf{c}$ (Figure 8), the balance coefficient is optimal for this design and the $\hat{J}_2$ value was the lowest obtained from 20 trials of each type of algorithm.

A comparison can be made among the three efficient designs together with the full factorial (Table 2). The full factorial design (D_4) is clearly balanced, but it requires all 105 design points. As the number of runs increases, the values of standardized balance coefficient and standardized J_2-optimality both decrease.

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 7 \\ 1 & 3 & 4 \\ 1 & 4 & 2 \\ 1 & 5 & 6 \\ 1 & 1 & 3 \\ 1 & 2 & 6 \\ 1 & 3 & 1 \\ 1 & 4 & 5 \\ 1 & 5 & 2 \\ 2 & 1 & 4 \\ 2 & 2 & 5 \\ 2 & 3 & 2 \\ 2 & 4 & 7 \\ 2 & 5 & 1 \\ 2 & 1 & 7 \\ 2 & 2 & 3 \\ 2 & 3 & 6 \\ 2 & 4 & 3 \\ 2 & 5 & 4 \\ 3 & 1 & 2 \\ 3 & 2 & 2 \\ 3 & 3 & 5 \\ 3 & 4 & 1 \\ 3 & 5 & 7 \\ 3 & 1 & 6 \\ 3 & 2 & 1 \\ 3 & 3 & 3 \\ 3 & 4 & 4 \\ 3 & 5 & 5 \end{bmatrix} \qquad [\,\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3\,] = \begin{bmatrix} 10 & 6 & 5 \\ 10 & 6 & 5 \\ 10 & 6 & 4 \\ & 6 & 4 \\ & 6 & 4 \\ & & 4 \\ & & 4 \end{bmatrix}$$

FIGURE 8. $EA(30,\ 3^1 5^1 7^1)$ and Its Factor Level Count Vectors.

Summary

Mixed-level factorial designs are necessary alternatives to traditional two-level factorial designs when qualitative factors are involved. With the exception of computer-generated designs, only balanced orthogonal and near-orthogonal designs have been pre-

TABLE 2. Comparison of *EA*s with the Full Factorial

	Design	No of runs	$\hat{H}$	$\hat{J}_2$
D1	$EA(15, 3^1 5^1 7^1)$	15	0.00130	0.0038
D2	$EA(21, 3^1 5^1 7^1)$	21	0.00060	0.0033
D3	$EA(30, 3^1 5^1 7^1)$	30	0.00053	0.0026
D4	Full Factorial $3^1 5^1 7^1$	105	0.00000	0.0009

viously addressed in the literature. For the myriad of situations when balance cannot be obtained, near-balanced, near-orthogonal designs are a reasonable choice. This paper proposes a criterion called a balance coefficient to assess relative deviations from perfect balance and combines it with a modified orthogonality criterion into an objective function for design selection. Designs are generated and compared using three methods: the GA, the coordinate exchange algorithm, and the columnwise–pairwise algorithm. These efficient mixed-level fractional designs are a possible solution to the excessive run requirements associated with many balanced mixed-level fractional factorial designs.

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Appendix
Efficient Mixed-Level Designs

TABLE A1. $EA(20, 2^4 3^1 4^1)$

1	1	1	2	1	4
1	2	1	2	3	1
1	1	1	1	1	4
1	2	2	1	1	2
1	1	2	1	3	4
1	2	2	2	2	3
1	1	2	2	1	3
1	2	1	1	2	3
1	1	1	1	2	1
1	2	2	2	3	2
2	1	2	2	2	2
2	2	2	1	2	4
2	1	2	1	3	3
2	2	2	2	1	1
2	1	1	2	2	2
2	2	1	2	2	4
2	1	2	1	1	1
2	2	1	1	1	2
2	1	1	2	3	1
2	2	1	1	3	3

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$	$\mathbf{c}_5$	$\mathbf{c}_6$
10	10	10	10	7	5
10	10	10	10	7	5
				6	5
					5

TABLE A2. $EA(20, 2^4 3^1 5^1)$

1	1	1	2	2	3
1	2	1	1	1	1
1	1	2	2	3	1
1	2	1	1	3	2
1	1	2	1	2	2
1	2	2	2	1	5
1	1	1	2	3	4
1	2	2	2	2	4
1	1	2	1	1	3
1	2	1	1	2	5
2	1	1	1	2	1
2	2	2	1	3	4
2	1	2	2	1	2
2	2	2	1	3	3
2	1	1	1	1	4
2	2	1	2	1	3
2	1	2	1	2	5
2	2	2	2	2	1
2	1	1	2	3	5
2	2	1	2	1	2

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$	$\mathbf{c}_5$	$\mathbf{c}_6$
10	10	10	10	7	4
10	10	10	10	7	4
				6	4
					4
					4

TABLE A3. $EA(15, 2^1 3^1 5^1 7^1)$

1	1	4	4
1	2	5	7
1	3	3	6
1	1	2	1
1	2	1	3
1	3	5	1
1	1	3	5
2	2	4	6
2	3	2	5
2	1	1	7
2	2	3	1
2	3	1	4
2	1	5	3
2	2	2	2
1	3	4	2
$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$

$$\begin{bmatrix} 8 & 5 & 3 & 3 \\ 7 & 5 & 3 & 2 \\ & 5 & 3 & 2 \\ & & 3 & 2 \\ & & 3 & 2 \\ & & & 2 \\ & & & 2 \end{bmatrix}$$

TABLE A4. $EA(21, 2^6 3^1 5^1 7^1)$

$$\begin{bmatrix}
1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 4 \\
1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 7 \\
1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 & 1 & 2 & 1 & 4 & 5 \\
1 & 1 & 2 & 2 & 1 & 2 & 3 & 5 & 1 \\
1 & 2 & 2 & 2 & 1 & 1 & 1 & 3 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 4 & 6 \\
1 & 2 & 1 & 1 & 1 & 1 & 3 & 1 & 5 \\
1 & 1 & 2 & 1 & 2 & 1 & 3 & 1 & 3 \\
1 & 2 & 1 & 1 & 2 & 2 & 1 & 5 & 2 \\
2 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 3 \\
2 & 2 & 1 & 1 & 2 & 2 & 3 & 4 & 4 \\
2 & 1 & 2 & 1 & 1 & 2 & 3 & 3 & 2 \\
2 & 2 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 6 \\
2 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\
2 & 1 & 2 & 2 & 2 & 1 & 1 & 4 & 7 \\
2 & 2 & 2 & 2 & 2 & 1 & 3 & 2 & 6 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 5 & 5 \\
2 & 2 & 1 & 2 & 1 & 1 & 3 & 5 & 7 \\
1 & 1 & 1 & 1 & 2 & 1 & 1 & 3 & 1
\end{bmatrix}$$

$\mathbf{c}_1$ $\mathbf{c}_2$ $\mathbf{c}_3$ $\mathbf{c}_4$ $\mathbf{c}_5$ $\mathbf{c}_6$ $\mathbf{c}_7$ $\mathbf{c}_8$ $\mathbf{c}_9$

$$\begin{bmatrix}
11 & 11 & 11 & 11 & 11 & 7 & 5 & 3 \\
10 & 10 & 10 & 10 & 10 & 7 & 4 & 3 \\
 & & & & & 7 & 4 & 3 \\
 & & & & & 7 & 4 & 3 \\
 & & & & & & 4 & 3 \\
 & & & & & & 4 & 3 \\
 & & & & & & & 3 \\
 & & & & & & & 3
\end{bmatrix}$$

TABLE A5. $EA(30, 2^6 3^1 5^1 7^1)$

$$\begin{bmatrix}
1 & 1 & 2 & 2 & 1 & 2 & 3 & 3 & 1 \\
1 & 2 & 2 & 1 & 2 & 1 & 3 & 1 & 3 \\
1 & 1 & 1 & 2 & 2 & 1 & 3 & 4 & 4 \\
1 & 2 & 1 & 1 & 1 & 1 & 1 & 3 & 7 \\
1 & 1 & 2 & 1 & 1 & 1 & 1 & 4 & 5 \\
1 & 2 & 1 & 1 & 2 & 1 & 3 & 5 & 5 \\
1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 4 \\
1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\
1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 2 \\
1 & 2 & 1 & 2 & 1 & 1 & 3 & 2 & 1 \\
1 & 1 & 1 & 2 & 2 & 1 & 2 & 3 & 6 \\
1 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 6 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 5 & 3 \\
1 & 2 & 2 & 2 & 1 & 2 & 2 & 4 & 7 \\
1 & 1 & 2 & 1 & 2 & 2 & 1 & 5 & 1 \\
2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\
2 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 2 \\
2 & 2 & 2 & 1 & 2 & 1 & 2 & 4 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 3 & 2 & 6 \\
2 & 2 & 1 & 1 & 2 & 2 & 2 & 3 & 4 \\
2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 7 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 5 \\
2 & 1 & 2 & 2 & 2 & 2 & 3 & 1 & 7 \\
2 & 2 & 2 & 2 & 1 & 2 & 3 & 5 & 6 \\
2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 5 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 5 & 4 \\
2 & 1 & 2 & 2 & 2 & 1 & 1 & 3 & 3 \\
2 & 2 & 1 & 2 & 1 & 2 & 1 & 4 & 3 \\
2 & 2 & 2 & 1 & 1 & 2 & 3 & 3 & 2
\end{bmatrix}$$

$\mathbf{c}_1$ $\mathbf{c}_2$ $\mathbf{c}_3$ $\mathbf{c}_4$ $\mathbf{c}_5$ $\mathbf{c}_6$ $\mathbf{c}_7$ $\mathbf{c}_8$ $\mathbf{c}_9$

$$\begin{bmatrix}
15 & 15 & 15 & 15 & 15 & 10 & 6 & 5 \\
15 & 15 & 15 & 15 & 15 & 10 & 6 & 5 \\
 & & & & & 10 & 6 & 4 \\
 & & & & & 7 & 4 & 3 \\
 & & & & & & 6 & 4 \\
 & & & & & & 6 & 4 \\
 & & & & & & & 4 \\
 & & & & & & & 4
\end{bmatrix}$$

TABLE A6. *EA*(21, $3^2 5^1 7^1$)

1	1	2	4
1	2	1	5
1	3	1	2
1	1	3	7
1	2	5	6
1	3	4	1
1	1	4	3
2	2	4	4
2	3	3	5
2	1	1	6
2	2	5	7
2	3	3	3
2	1	1	1
2	2	2	2
3	3	1	7
3	1	4	5
3	2	3	1
3	3	2	6
3	1	5	2
3	2	2	3
3	3	5	4

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$
7	7	5	3
7	7	4	3
7	7	4	3
		4	3
		4	3
			3
			3

TABLE A7. *EA*(21, $3^1 4^1 7^1$)

1	1	6
1	2	3
1	3	1
1	4	5
1	1	2
1	2	7
1	3	4
2	4	4
2	1	3
2	2	5
2	3	2
2	4	6
2	1	7
2	2	1
3	3	5
3	4	3
3	1	1
3	2	2
3	3	6
3	4	7
3	1	4

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3 4$
7	6	3
7	5	3
7	5	3
	5	3
		3
		3
		3

TABLE A8. $EA(21, 3^2 4^1 7^1)$

1	1	4	7
1	2	1	4
1	3	3	6
1	1	1	5
1	2	2	1
1	3	3	3
1	1	2	2
2	2	4	2
2	3	4	4
2	1	1	3
2	2	1	6
2	3	2	5
2	1	3	1
2	2	3	7
3	3	1	1
3	1	3	4
3	2	2	3
3	3	2	7
3	1	4	6
3	2	4	5
3	3	1	2

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$
7	7	6	3
7	7	5	3
7	7	5	3
		5	3
		5	3
			3
			3
			3

TABLE A9. $EA(20, 2^3 5^1 7^1)$

1	1	2	4	4
1	2	1	5	3
1	1	2	5	2
1	2	1	2	5
1	1	2	3	5
1	2	2	3	1
1	1	2	2	6
1	2	1	1	4
1	1	1	1	7
1	2	1	4	6
2	1	1	5	4
2	2	1	3	2
2	1	1	2	2
2	2	2	5	5
2	1	1	4	1
2	2	2	4	3
2	1	1	3	3
2	2	2	1	6
2	1	2	1	1
2	2	2	2	7

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$	$\mathbf{c}_5$
10	10	10	4	3
10	10	10	4	3
			4	3
			4	3
			4	3
				3
				2

TABLE A10. *EA*(20, $2^4 5^1 7^1$)

1	1	1	1	1	2
1	2	2	1	4	3
1	1	2	1	2	1
1	2	1	2	4	4
1	1	2	2	5	5
1	2	1	2	3	1
1	1	1	2	1	3
1	2	1	1	5	7
1	1	2	2	2	6
1	2	2	1	3	6
2	1	2	2	4	7
2	2	2	2	1	1
2	1	1	1	4	5
2	2	2	2	5	2
2	1	1	2	3	2
2	2	1	2	2	3
2	1	2	1	3	4
2	2	2	1	1	4
2	1	1	1	5	6
2	2	1	1	2	5

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$	$\mathbf{c}_5$	$\mathbf{c}_6$
10	10	10	10	4	3
10	10	10	10	4	3
				4	3
				4	3
				4	3
					3
					2

TABLE A11. *EA*(20, $3^1 4^1 5^1$)

3	1	1
2	1	2
1	1	3
1	1	4
2	1	5
1	2	1
3	2	2
2	2	3
1	2	4
2	2	5
2	3	1
3	3	2
1	3	3
2	3	4
3	3	5
2	4	1
1	4	2
3	4	3
3	4	4
1	4	5

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$4
7	5	4
7	5	4
6	5	4
	5	4
		4

TABLE A12. $EA(20, 2^3 3^1 4^1 5^1)$

1	1	2	1	2	2
1	2	1	1	1	2
1	1	2	2	3	5
1	2	2	3	4	1
1	1	1	1	3	1
1	2	2	2	3	3
1	1	1	3	2	4
1	2	2	2	1	4
1	1	1	3	4	3
1	2	1	2	2	5
2	1	2	3	1	5
2	2	1	3	3	2
2	1	2	1	1	1
2	2	1	1	4	5
2	1	2	2	4	2
2	2	2	1	2	3
2	1	1	1	4	4
2	2	2	3	3	4
2	1	1	2	1	3
2	2	1	2	2	1

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$	$\mathbf{c}_5$	$\mathbf{c}_6$
10	10	10	7	5	4
10	10	10	7	5	4
			6	5	4
				5	4
					4

TABLE A13. $EA(20, 2^4 3^1 4^1 5^1)$

1	1	2	2	2	3	3
1	2	1	2	1	1	3
1	1	2	1	3	1	2
1	2	1	2	3	4	4
1	1	2	1	1	4	1
1	2	1	1	2	3	2
1	1	1	2	2	4	5
1	2	2	1	3	2	5
1	1	2	2	1	2	4
1	2	1	1	1	2	1
2	1	2	2	2	2	2
2	2	2	1	1	3	4
2	1	1	1	3	2	3
2	2	2	1	2	4	3
2	1	1	2	3	3	1
2	2	2	2	3	3	5
2	1	1	1	1	1	5
2	2	2	2	2	1	1
2	1	1	1	2	1	4
2	2	1	2	1	4	2

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$	$\mathbf{c}_5$	$\mathbf{c}_6$	$\mathbf{c}_7$
10	10	10	10	7	5	4
10	10	10	10	7	5	4
				6	5	4
					5	4
						4

TABLE A14. $EA(28, 2^3 6^1 7^1)$

1	1	1	2	2
1	2	2	4	3
1	1	1	4	6
1	2	1	3	7
1	1	1	6	1
1	2	2	6	3
1	1	2	5	5
1	2	1	2	4
1	1	1	1	7
1	2	2	3	2
1	1	2	3	4
1	2	2	2	1
1	1	2	1	5
1	2	1	5	6
2	1	2	4	2
2	2	1	6	5
2	1	2	3	1
2	2	2	1	6
2	1	1	5	4
2	2	2	4	7
2	1	1	2	3
2	2	1	4	5
2	1	2	6	6
2	2	2	1	4
2	1	1	3	3
2	2	1	5	2
2	1	2	2	7
2	2	1	1	1

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$	$\mathbf{c}_5$
14	14	14	5	4
14	14	14	5	4
			5	4
			5	4
			4	4
			4	4
				4

TABLE A15. $EA(28, 2^4 6^1 7^1)$

1	1	2	2	5	7
1	2	2	1	6	3
1	1	2	2	2	6
1	2	1	2	4	4
1	1	2	2	1	3
1	2	1	1	3	5
1	1	2	1	1	5
1	2	2	2	3	4
1	1	1	1	4	1
1	2	1	2	2	7
1	1	1	1	2	2
1	2	2	1	6	1
1	1	1	2	3	2
1	2	1	1	4	6
2	1	1	1	2	3
2	2	1	1	1	7
2	1	2	1	4	7
2	2	1	2	1	1
2	1	1	1	5	4
2	2	2	1	2	3
2	2	2	2	2	5
2	1	2	1	1	4
2	2	2	2	4	2
2	1	1	2	6	6
2	2	2	1	5	2
2	1	2	2	3	1
2	2	2	1	3	6
2	1	1	2	6	5
2	2	1	2	5	3

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$	$\mathbf{c}_5$	$\mathbf{c}_6$
14	14	14	14	5	4
14	14	14	14	5	4
				5	4
				5	4
				4	4
				4	4
					4

TABLE A16. $EA(21, 3^1 6^1 7^1)$

1	1	4
1	2	5
1	3	1
1	4	6
1	5	3
1	6	2
1	1	7
2	2	2
2	3	7
2	4	3
2	5	5
2	6	1
2	1	6
2	2	4
3	3	2
3	4	5
3	5	4
3	6	7
3	1	3
3	2	1
3	3	6

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$
7	4	3
7	4	3
7	4	3
	3	3
	3	3
	3	3
		3

TABLE A17. $EA(24, 4^1 6^1 7^1)$

1	1	1
1	2	3
1	3	7
1	4	5
1	5	6
2	1	6
2	2	1
2	3	4
2	4	7
2	5	2
2	6	3
3	1	4
3	2	2
3	3	3
3	4	1
3	5	5
3	6	7
4	1	2
4	2	6
4	3	1
4	4	4
4	5	3
4	6	5

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$
6	4	4
6	4	4
6	4	4
6	4	3
	4	3
	4	3
		3
		3

TABLE A18. $EA(24, 5^1 6^1 7^1)$

1	1	6
2	1	3
3	1	2
4	1	7
5	2	2
1	2	1
2	2	7
3	2	3
4	3	2
5	3	6
1	3	3
2	3	5
3	4	1
4	4	5
5	4	4
1	4	2
2	5	4
3	5	5
4	5	1
5	5	3
1	6	4
2	6	1
3	6	7
4	6	6

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$
5	4	4
5	4	4
5	4	4
5	4	3
4	4	3
	4	3
		3

TABLE A19. $EA(24, 3^1 5^1 6^1 7^1)$

1	4	1	3
1	1	2	7
1	5	3	5
1	3	4	6
1	1	5	3
1	4	6	4
1	2	1	1
1	3	2	2
2	1	3	2
2	4	4	2
2	2	5	7
2	3	6	1
2	1	1	4
2	4	2	5
2	2	3	6
2	5	4	3
3	3	5	5
3	1	6	6
3	5	1	7
3	5	2	1
3	3	3	3
3	2	4	4
3	4	5	1
3	2	6	2

$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$
8	5	4	4
8	5	4	4
8	5	4	4
	5	4	3
	4	4	3
		4	3
			3

TABLE A20. $EA(20, 4^1 5^1 6^1 7^1)$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 5 & 6 \\ 1 & 3 & 6 & 7 \\ 1 & 4 & 3 & 5 \\ 1 & 5 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 2 & 2 & 6 & 5 \\ 2 & 3 & 4 & 6 \\ 2 & 4 & 2 & 2 \\ 2 & 5 & 3 & 1 \\ 3 & 1 & 4 & 2 \\ 3 & 2 & 2 & 1 \\ 3 & 3 & 1 & 3 \\ 3 & 4 & 5 & 4 \\ 3 & 5 & 6 & 6 \\ 4 & 1 & 4 & 7 \\ 4 & 2 & 3 & 3 \\ 4 & 3 & 2 & 4 \\ 4 & 4 & 1 & 1 \\ 4 & 5 & 4 & 5 \end{bmatrix}$$

$$\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4$$

$$\begin{bmatrix} 5 & 4 & 4 & 3 \\ 5 & 4 & 4 & 3 \\ 5 & 4 & 3 & 3 \\ 5 & 4 & 3 & 3 \\ & 4 & 3 & 3 \\ & & 3 & 3 \\ & & & 2 \end{bmatrix}$$

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