

# Linear Algebra

## *Vectors and Matrices*

This is an elementary introduction to vectors and matrices.

A vector is simply an array of numbers written as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad (1)$$

where a matrix is written as a collection of arrays, i.e.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad (2)$$

In (1), we have a single column of 3 numbers, whereas in (2) we have two rows of 3 numbers in each row. In general we could have a vector of  $m$  elements written as

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

are a matrix of  $m$  rows and  $n$  columns in which we would have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

where  $a_{mn}$  is the element in the  $m^{\text{th}}$  row and  $n^{\text{th}}$  column.

For those that have seen vectors in either Physics or maybe Calc 3, another way to write them is

$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$

A natural question is: "Can we add, subtract and multiply both vectors and matrices?"

### *Vector Addition*

To add vectors, we simply add component by component. So, if

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

then  $\vec{u} + \vec{v}$  becomes

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 2+5 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Subtraction follows similarly,  $\vec{u} - \vec{v}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1-4 \\ 2-5 \\ 3-6 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

### *Vector Scalar Multiplication*

To scale a vector, we simply multiply each component by the scale factor. So,

$$6\vec{u} = 6 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} \quad -5\vec{v} = -5 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -20 \\ -25 \\ -30 \end{bmatrix}$$

we can also perform both simultaneously,  $a\vec{u} + b\vec{v}$  would produce

$$\begin{bmatrix} a + 4b \\ 2a + 5b \\ 3a + 6b \end{bmatrix},$$

### *Matrix Addition(Subtraction)*

To add matrices, again, we simply add(subtract) component by component. So, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 5 & 2 \\ 4 & -3 & 9 \end{bmatrix}$$

then  $A + B$  would produce

$$\begin{bmatrix} 1+(-1) & 2+5 & 3+2 \\ 4+4 & 5-3 & 6+9 \end{bmatrix} = \begin{bmatrix} 0 & 7 & 5 \\ 8 & 2 & 15 \end{bmatrix},$$

whereas  $A - B$  would produce

$$\begin{bmatrix} 1 - (-1) & 2 - 5 & 3 - 2 \\ 4 - 4 & 5 - (-3) & 6 - 9 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 8 & -3 \end{bmatrix}.$$

### *Matrix Scalar Multiplication*

For matrices, we simply multiply each component by the scale factor. So,

$$-2A = -2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \end{bmatrix}$$

### *Matrix Multiplication*

In order to multiply two matrices, it is important that the size of the matrices be of a certain dimension. For example, if

$$A_{mn} B_{kl},$$

then we require that  $n = k$  and the dimension of the new matrix is  $m \times l$ . Consider the  $2 \times 2$  matrix and the  $2 \times 3$  matrix

$$A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

As  $A_{22}$  and  $B_{23}$  we can multiply  $AB$  but not  $BA$ . Thus, the command

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 & 1 \cdot 3 + 2 \cdot 4 & 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 1 + 4 \cdot 2 & 3 \cdot 3 + 4 \cdot 4 & 3 \cdot 5 + 4 \cdot 6 \end{bmatrix}$$

giving

$$AB = \begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \end{bmatrix}$$

## Determinants

To calculate determinants matrices must be square *i.e.* an  $n \times n$  matrix. For  $2 \times 2$  if  $A$  is given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the determinant of  $A$  denoted by  $\det A$  is given by

$$\det A = ad - bc$$

## Inverses

Given an  $n \times n$  matrix, if a second matrix  $B$  exists such that  $AB = I$ , the identity matrix.

The identity matrix is given by

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

If the matrix  $B$  exists, it's called the inverse. Often this is denoted by  $A^{-1}$ . For  $2 \times 2$  matrices, this is given by

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For example, if

$$A = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ -1 & 5 \end{bmatrix}$$

A quick calculation show that

$$\begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 1 & -2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

## *Eigenvalues and Eigenvectors*

Given a square matrix  $A$ , if a vector  $\vec{v}$  and scalar  $\lambda$  exists such that

$$A\vec{v} = \lambda\vec{v}$$

then these are called eigenvector and eigenvalues. In order to calculate the eigenvalues, it is necessary to calculate

$$\det|A - \lambda I| = 0 \tag{3}$$

This will give rise to a polynomial for  $\lambda$ . For a given  $\lambda$ , (3) will give rise to the associated eigenvector. Consider the following example. If

$$A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$$

then

$$\det|A - \lambda I| = \begin{vmatrix} 2 - \lambda & 2 \\ 5 & -1 - \lambda \end{vmatrix} = (\lambda + 3)(\lambda - 4) = 0$$

giving rise to the eigenvalues  $\lambda = -3, 4$ .

If  $\lambda = -3$  then

$$\begin{bmatrix} 2 - (-3) & 2 \\ 5 & -1 - (-3) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

or

$$\begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Thus, one possible eigenvector is

$$\vec{u} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

If  $\lambda = 4$  then

$$\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Thus, one possible eigenvector is

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$