Math 6345 Advanced ODEs
Homework 2

1. Find e^{At} by (1) using the fundamental matrix method and (2) using the series definition for the following

$$(i)\ \ A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad (ii)\ \ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

2. Solve

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \bar{x}, \qquad \bar{x}(0) = \bar{x}_0$$

by calculating e^{At}.

Hint. Show that matrix can be written as

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix},$$

and the fact that since they commute we can use the relation $e^{(A+B)t} = e^{At}e^{Bt}$.

3. Some properties of the matrix exponential
(i) If there exists a nonsingular matrix T such that

$$A = TDT^{-1},$$

where D is diagonal, prove that

$$e^{At} = Te^{Dt}T^{-1}$$

(ii) Prove

$$(e^{A})^{-1} = e^{-A}$$

(iii) Prove

$$det(e^{A}) = e^{trA}$$

(iv) If A is a 2×2 matrix with a repeated eigenvalue of r, show that

$$e^{At} = e^{rt}\left[\mathbb{I} + (A - r\mathbb{I})t\right].$$

4. If $A(t)$ is an 2×2 matrix of continuous functions on $I = [a, b]$ and $\Phi(t)$ a matrix of differentiable functions such that

$$\Phi'(t) = A(t)\Phi(t),$$

prove that for $t, t_0 \in I$, then

$$\det \Phi(t) = \det \Phi(t_0) e^{\int_{t_0}^{t} trA(s)ds}$$