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ON THE FUNCTIONS $\varphi_r(\mathrm{n})$ AND $\rho_r(\mathrm{n})$

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Abstract

In this paper we find the average orders of the functions $\varphi_r(\mathrm{n})$ *and* $\rho_r(\mathrm{n})$(defined in the Introduction). *Also since* $\varphi_r(\mathrm{n})<\rho_r(\mathrm{n})$, *to compare the growth rates of these functions we obtain the average orders of* $\frac{\rho_r(\mathrm{n})}{\varphi_r(\mathrm{n})}$ *and* $\frac{\varphi_r(\mathrm{n})}{\rho_r(\mathrm{n})}$. *Which generalize these results proved for* $\varphi(\mathrm{n})$ *and* $\rho(\mathrm{n})$ *by Laszlo Toth [4].*

Keywords: *r*-regular integer modulo n^r, *r*-free integer, *r*-gcd of two integers.

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1. INTRODUCTION:

Let *r* be a fixed positive integer. A positive integer *a* is said to be *r-regular modulo* n^r if there is an integer *x* such that $a^{r+1}x \equiv a^r \left(\bmod n^r\right)$. The case $r=1$ gives the notion of *a regular integer modulo n*, introduced by J. Morgado ([6] and [7]) who made an investigation of their properties.

Clearly $a=0$ is *r*-regular modulo n^r for every $n \geq 1$. Also if $a \equiv b\left(\bmod n^r\right)$ then *a* and *b* are *r*-regular modulo n^r simultaneously. Further, if *a* and *b* are *r*-regular modulo n^r then so is *ab*.

For positive integers *a* and *b* their greatest r^{th} power common divisor is denoted by $(a,\ b)_r$ and is called the *r-gcd of a and b*. Note that $(a,\ b)_1=(a,\ b)$, the gcd of a and *b*.

We recall the notions given in ([5], *p* 42–43):

A complete set of residues modulo n^r is called a $(n,\ r)$*-residue system*. $C_{n,\ r}=\left\{a:1\leq a\leq n^r\right\}$ is the *minimal* $(n,\ r)$*-residue system*. The set of all *a* in an $(n,\ r)$-residue system such that $\left(a,\ n^r\right)_r=1$ is called *a reduced* $(n,\ r)$*-residue system*. $R_{n,\ r}=\left\{a\in C_{n,\ r}\ :\ \left(a,\ n^r\right)_r=1\right\}$ is the *minimal reduced* $(n,\ r)$*-residue system.*

V.L. Klee [3] defined a generalization φ_r of the Euler's function by $\varphi_r(\mathrm{n})=\# R_{n,r}$

Let $\mathrm{Reg}_r(n)=\{a\in C_{n,r}: a \text{ is r-regular modulo } n^r\}$ and $\rho_r(n)=\#\mathrm{Re}\,g_r(n)$

Observe that any $a\in R_{n,r}$ is in $\mathrm{Reg}_r(n)$. In fact, if $a\in R_{n,r}$ then $(a, n^r)_r=1$ so that $(a, n^r)=1$ and therefore there is an integer x_0 such that $a\,x_0\equiv 1\left(\bmod n^r\right)$ which gives $a^{r+1}x_0\equiv a^r\left(\bmod n^r\right)$ showing $a\in\mathrm{Reg}_r(n)$. Hence

(1.1) $\phi_r(n)<\rho_r(n)\le n^r$ for every $n>1$ and $r\ge 1$, with $\rho_r(n)=n^r$ if and only if n is squarefree.

Recently Laszlo Toth [4] has studied several properties of the function $\rho(n):=\rho_1(n)$.

The purpose of this paper is to first obtain the average order of $\rho_r(n)$ in section 2. Also in view of (1.1), to compare the growth rate of the functions $\rho_r(n)$ and $\varphi_r(n)$, we investigate the average order of the function $\dfrac{\rho_r(n)}{\varphi_r(n)}$ in section 3 and that of $\dfrac{\varphi_r(n)}{\rho_r(n)}$ in section 4. In the process we obtain an asymptotic formula for the summatory function of $\varphi_r(n)$ and thereby its average order in section 4.

2. AVERAGE ORDER OF $\rho_r(n)$

Eckford Cohen [2] called a divisor d of a positive integer n unitary if $\gcd\left(d,\frac{n}{d}\right)=1$. In this case one writes $d\,||\,n$.

Note that $\rho_r(n)$ is a multiplicative function of n and that

(2.1) $$\rho_r(n)=\sum_{d\|n}\varphi_r(d),$$

where the sum is over all unitary divisors d of n, so that for any prime p and integer $\alpha\ge 1$ we have

(2.2) $$\rho_r(p^\alpha)=\varphi_r(p^\alpha)+1=p^{\alpha r}-p^{(\alpha-1)r}+1.$$

In a very recent paper Bradut Apostol and László Tóth (*Some remarks on regular integers modulo* n, see http://arXiv.org/abs/1304.2699v1[math.NT] 9 April 2013) have introduced multidimensional generalization $V_r(n)$ of the function $\rho(n)$ as follows:

2.3. Definition. For any $r\ge 1$, let $V_r(n)$ denote the number of order r-tuples $(m_1,m_2,\dots, m_r)\in\{1,2,3,\dots,n\}^r$ such that $\gcd(m_1,m_2,\dots, m_r)$ is regular modulo n.

Note that $V_1(n)=\rho(n)$.

In a study of the function they have shown that

(2.4) $$V_r(n)=\sum_{d\|n}\varphi_r(d).$$

Also from one of their results (Proposition 2 of the above quoted paper) the lemma given below follows:

2.5. Lemma. For $r>1$,

$$\sum_{n\le x}V_r(n)=A_r\cdot\frac{x^{r+1}}{r+1}+O\left(x^r\right),$$

where

(2.6) $$A_r=\prod_p\left\{1-\frac{1}{p^{r+1}}\frac{(p-1)}{p\left(p^{r+1}-1\right)}\right\},$$

in which the product is over all primes p.

In view of (2.1) and (2.4), it is obvious that $\rho_r(n) = V_r(n)$ for integers $r \geq 1$ and $n \geq 1$. Therefore it follows from Lemma 2.5 that

(2.7) $$\sum_{n\leq x} \rho_r(n) = A_r \cdot \frac{x^{r+1}}{r+1} + O(x^r),$$

from which we find that the average order of $\rho_r(n)$ is $\frac{A_r x^r}{r+1}$.

3. AVERAGE ORDER OF $\rho_r(\mathrm{n})/\varphi_r(\mathrm{n})$

We begin with

3.1. Lemma. The Dirichlet series

$\sum_{n=1}^{\infty} \frac{\varphi(\mathrm{n})}{\varphi_r(\mathrm{n})n^2}$ converges to B_r where

$$B_r = \prod_p \left\{1 + \frac{p^{r-1}(p-1)}{(p^r-1)(p^{r+1}-1)}\right\}$$

Note that $B_1 = \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = \zeta(2) = \frac{\pi^2}{6}$.

Proof :- Since

$$(3.1)' \qquad \frac{\varphi(n)}{\varphi_r(n)n^2} = \frac{n\prod_{p|n}\left(1-\frac{1}{p}\right)}{n^r \prod_{p|n}\left(1-\frac{1}{p^r}\right).n^2} = \frac{1}{n^{r+1}} \prod_{p|n}\left(\frac{1}{1+\frac{1}{p}+\ldots+\frac{1}{p^{r-1}}}\right) < \frac{1}{n^{r+1}},$$

the series is convergent.

By Euler product representation of Dirichlet series ([1], Theorem 11.6), we get

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{\varphi_r(n)n^2} = \prod_p \left\{1 + \sum_{j=1}^{\infty} \frac{\varphi(p^j)}{\varphi_r(p^j)p^{2j}}\right\}$$

$$= \prod_p \left\{1 + \sum_{j=1}^{\infty} \frac{p^{j-1}(p-1)}{p^{(j-1)r}(p^r-1)p^{2j}}\right\}$$

$$= \prod_p \left\{1 + \frac{(p-1)p^{r-1}}{(p^r-1)} \sum_{j=1}^{\infty} \frac{1}{p^{j(r+1)}}\right\}$$

$$= \prod_p \left\{1 + \frac{(p-1)p^{r-1}}{(p^r-1)} . \frac{1}{p^{r+1}} \frac{1}{\left(1-\frac{1}{p^{r+1}}\right)}\right\}$$

$$= \prod_p \left\{1 + \frac{p^{r-1}(p-1)}{(p^r-1)(p^{r+1}-1)}\right\} = B_r,$$

proving the lemma.

3.2 Lemma. For $x \geq 2$ and $r \geq 1$,

$$\sum_{n>x} \frac{\varphi(\mathrm{n})}{\varphi_r(\mathrm{n})n^2} = O\left(\frac{1}{x^r}\right)$$

Proof :- In view of $(3.1)'$, we get

$$\sum_{n>x} \frac{\varphi(\mathrm{n})}{\varphi_r(\mathrm{n})n^2} < \sum_{n>x} \frac{1}{n^{r+1}} = O\left(\frac{1}{x^r}\right)$$

3.3. For any $x \geq 1$, if $\varphi(x;\, n)$ is the number of positive integers m such that $m \leq x$ and $(m,n)=1$, it is well-known that (see [1], problem 9, p. 47)

$$\varphi(x;\, n) := \sum_{\substack{m \leq x \\ (m,n)=1}} 1 = \sum_{d|n} \mu(d) \left[\frac{x}{d}\right],$$ where $[y]$ is the greatest integer not exceeding y.

Now using that $[y] = y + O(1)$ in the above it is easy to prove that

(3.4) $$\varphi(x;\, n) = \frac{\varphi(n)}{n} x + O(\theta(n)),$$

where $\theta(n)$ is the number of squarefree divisors of n. It may be noted that $\theta(n)$ is also equal to the number of unitary divisors of n and that $\theta(n) = 2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime divisors of n.

3.5. Theorem. For $x \geq 2$

$$\sum_{n \leq x} \frac{\rho_r(n)}{\varphi_r(n)} = B_r . x + O\left(\frac{\log^2 x}{x^{r-1}}\right),$$

where B_r is given in Lemma 3.1. That is, the average order of $\frac{\rho_r(n)}{\varphi_r(n)}$ is B_r.

Proof :- We know that

$$\rho_r(n) = \sum_{d\|n} \varphi_r(d) = \sum_{d\|n} \varphi_r\left(\frac{n}{d}\right) = \sum_{d\|n} \frac{\varphi_r(n)}{\varphi_r(d)}$$

from which we get

(3.6) $$\frac{\rho_r(n)}{\varphi_r(n)} = \sum_{d\|n} \frac{1}{\varphi_r(d)} = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} \frac{1}{\varphi_r(d)}.$$

Therefore, by (3.6), (3.4) and Lemma 3.2, we get

(3.7) $$\sum_{n \leq x} \frac{\rho_r(n)}{\varphi_r(n)} = \sum_{\substack{d\delta \leq x \\ (d,\delta)=1}} \frac{1}{\varphi_r(d)}$$

$$= \sum_{d \leq x} \frac{1}{\varphi_r(d)} \left(\sum_{\substack{\delta \leq \frac{x}{d} \\ (\delta, d)=1}} 1 \right)$$

$$= \sum_{d\le x}\frac{1}{\varphi_r(d)}\varphi\left(\frac{x}{d};d\right)$$
$$= \sum_{d\le x}\frac{1}{\varphi_r(d)}\left\{\frac{\varphi(d)}{d}\cdot\frac{x}{d}+O(\theta(d))\right\}$$
$$= x\cdot\sum_{d\le x}\frac{\varphi(d)}{\varphi_r(d)\cdot d^2}+O\left(\sum_{d\le x}\frac{\theta(d)}{\varphi_r(d)}\right)$$
$$= x\cdot\left\{B_r-\sum_{n>x}\frac{\varphi(n)}{\varphi_r(n)\cdot n^2}\right\}+O\left(\sum_{n\le x}\frac{\theta(n)}{\varphi_r(n)}\right)$$
$$= x\cdot\left\{B_r+O\left(\frac{1}{x^r}\right)\right\}+O\left(\sum_{n\le x}\frac{\theta(n)}{\varphi_r(n)}\right)$$
$$= B_r x+O\left(\frac{1}{x^{r-1}}\right)+O\left(\sum_{n\le x}\frac{\theta(n)}{\varphi_r(n)}\right)$$

To estimate the last error term in (3.7), we first note that if $\sigma(n)$ is the sum of the positive divisor of n then for any prime p and integer $\alpha\ge 1$,

$$\varphi_r(p^\alpha)\sigma(p^\alpha)=\left(p^{\alpha r}-p^{(\alpha-1)r}\right)\left(1+p+p^2+\ldots+p^\alpha\right)$$
$$=p^{\alpha(r+1)}+p^{\alpha(r+1)-1}+p^{\alpha(r+1)-2}+\ldots+p^{\alpha r}-p^{(\alpha-1)r+\alpha}-p^{(\alpha-1)r+\alpha-1}-\ldots-p^{(\alpha-1)r}$$
$$=p^{\alpha(r+1)}+\left(p^{\alpha(r+1)-1}-p^{\alpha(r+1)-r}\right)+\left(p^{\alpha(r+1)-2}-p^{\alpha(r+1)-r-1}\right)+\ldots+\left(p^{\alpha r}-p^{(\alpha-1)r}\right)$$
$$\ge p^{\alpha(r+1)}$$

That is, $\varphi_r(p^\alpha)\sigma(p^\alpha)\ge(p^\alpha)^{r+1}$ for every prime p and integer $\alpha\ge 1$ so that, by the multiplicativity of these two functions, it follows that

$\varphi_r(n)\sigma(n)\ge n^{r+1}$ and hence

(3.8) $$\varphi_r(n)\ge\frac{n^{r+1}}{\sigma(n)}.$$

Therefore, $\dfrac{\theta(n)}{\varphi_r(n)}\le\dfrac{\tau(n)}{\varphi_r(n)}\le\dfrac{\tau(n)\cdot\sigma(n)}{n^{r+1}}$ so that

(3.9) $$\sum_{n\le x}\frac{\theta(n)}{\varphi_r(n)}\le\sum_{n\le x}\frac{\tau(n)\cdot\sigma(n)}{n^{r+1}}.$$

Now, by the well-known result of Ramanujan that

$$\sum_{n\le x}\tau(n).\sigma(n)=O(x^2\log x)$$

and the Abel's identity, we can show that

(3.10) $$\sum_{n\le x}\frac{\tau(n).\sigma(n)}{n^{r+1}}=O\left(\frac{\log^2 x}{x^{r-1}}\right)$$

which on substituting in (3.7) gives that

$$\sum_{n\le x}\frac{\rho_r(n)}{\varphi_r(n)}=B_r x+O\left(\frac{1}{x^{r-1}}\right)+O\left(\frac{\log^2 x}{x^{r-1}}\right)$$
$$=B_r x+O\left(\frac{\log^2 x}{x^{r-1}}\right),$$

proving the theorem.

3.11 Remark: The case $r=1$ of Theorem 3.5 gives that $\sum_{n\le x}\frac{\rho(n)}{\varphi(n)}=\frac{\pi^2 x}{6}+O(\log^2 x)$, a result proved by László Tóth ([4], Theorem 4).

4. AVERAGE ORDER OF $\varphi_r(\mathrm{n})/\rho_r(\mathrm{n})$

In the case $r=1$, László Tóth ([4], Theorem 4) has proved an asymptotic formula for $\sum_{n\le x}\frac{\varphi(n)}{\rho(n)}$.

Therefore we consider $r>1$ in the rest of this paper.

4.1 Lemma For $x\ge 2$,

$$\sum_{n\le x}\varphi_r(n)=\frac{x^{r+1}}{(r+1)\zeta(r+1)}+O(x^r)$$

Proof :- Since $\varphi_r(n)=\sum_{d\delta=n}\mu(d)\delta^r$, by Mobius inversion formula and T. M. Apostol ([1], Theorem 3.2) we get

$$\begin{aligned}\sum_{n\le x}\varphi_r(n)&=\sum_{d\delta\le x}\mu(d)\delta^r\\&=\sum_{d\le x}\mu(d)\left(\sum_{\delta\le\frac{x}{d}}\delta^r\right)\\&=\sum_{d\le x}\mu(d)\left\{\frac{\left(\frac{x}{d}\right)^{r+1}}{r+1}+O\left(\frac{x^r}{d^r}\right)\right\}\\&=\frac{x^{r+1}}{r+1}\sum_{d\le x}\frac{\mu(d)}{d^{r+1}}+O\left(x^r\sum_{d\le x}\frac{\mu(d)}{d^r}\right)\\&=\frac{x^{r+1}}{r+1}\left\{\frac{1}{\zeta(r+1)}+O\left(\frac{1}{x^r}\right)\right\}+O(x^r),\end{aligned}$$

since $\sum_{d\le x}\frac{|\mu(d)|}{d^r}$ is the partial sum of a convergent series. Therefore

$$\begin{aligned}\sum_{n\le x}\varphi_r(n)&=\frac{x^{r+1}}{(r+1)\zeta(r+1)}+O(x)+O(x^r)\\&=\frac{x^{r+1}}{(r+1)\zeta(r+1)}+O(x^r).\end{aligned}$$

4.2 Lemma For $x\ge 1$,

$$\sum_{n\le x}\frac{\varphi_r(n)}{n^r}=\frac{x}{\zeta(r+1)}+O(\log x)$$

Proof :- We use Abel's identity and Lemma 4.1. In fact, if $A_r(x)=\sum_{n\le x}\varphi_r(n)$ then, by Abel's identity,

$$\sum_{n\le x}\frac{\varphi_r(n)}{n^r}=\frac{A_r(x)}{x^r}-\frac{A_r(1)}{1^r}+r\int_1^x\frac{A_r(t)}{t^{r+1}}dt$$

$$= \frac{x}{(r+1)\zeta(r+1)} + O(1) - 1 + r\int_1^x \left\{ \frac{1}{(r+1)\zeta(r+1)} + O\left(\frac{1}{t}\right) \right\} dt$$
$$= \frac{x}{(r+1)\zeta(r+1)} + \frac{r}{(r+1)\zeta(r+1)}(x-1) + O(\log x) + O(1)$$
$$= \frac{x}{\zeta(r+1)} + O(\log x),$$

proving the lemma.

Let us write

(4.3) $f_r(n) = \dfrac{\varphi_r(n)}{\rho_r(n)}$ for $r > 1$ and $n \geq 1$.

4.4. Lemma If γ_r is the multiplicative arithmetic function such that $f_r = \dfrac{\varphi_r}{E_r} * \gamma_r$, where φ_r is the generalized Euler function and $E_r(n) = n^r$ for $n \geq 1$ then

$$\gamma_r = f_r * \left(\mu * \frac{u}{E_r}\right),$$

where $u(n) = 1$ for all n and μ is the Möbius function. Here, as usual, $*$ denotes the Dirichlet product of arithmetic functions.

Proof :- Since $\varphi_r = E_r * \mu$, we get

$$\left(\frac{\varphi_r}{E_r}\right)(n) = \frac{(E_r * \mu)(n)}{E_r(n)}$$
$$= \sum_{d|n} \frac{E_r(d)\cdot \mu\left(\frac{n}{d}\right)}{E_r(n)}$$
$$= \sum_{d|n} \frac{\mu\left(\frac{n}{d}\right)}{E_r\left(\frac{n}{d}\right)}$$
$$= \left(u * \frac{\mu}{E_r}\right)(n),$$

so that $f_r = \left(u * \dfrac{\mu}{E_r}\right) * \gamma_r$ and hence

(4.5) $$\gamma_r = f_r * \left(u * \frac{\mu}{E_r}\right)^{-1} = f_r * u^{-1} * \left(\frac{\mu}{E_r}\right)^{-1},$$

where the inverses of the arithmetic functions are with respect to the Dirichlet product. Also $u^{-1} = \mu$ and $\mu^{-1} = u$. Therefore the lemma follows from (4.5).

Now we study some more properties of the function γ_r in the next two lemmas.

4.6. Lemma. The function γ_r is such that for any prime p

(i) $\gamma_r(p) = 0$

(ii) $\gamma_r\left(p^2\right) < \frac{1}{p^r}$

and

(iii) $\left|\gamma_r\left(p^k\right)\right| \le \frac{1}{p^{k+1}}$ for any integer $k \ge 3$.

Proof :- By Lemma 4.4, we have

(4.7)
$$\gamma_r\left(p^k\right) = \left\{f_r * \left(\mu * \frac{u}{E_r}\right)\right\}\left(p^k\right)$$
$$= \sum_{j=0}^{\infty} f_r\left(p^j\right)\cdot\left(\mu * \frac{u}{E_r}\right)\left(p^{k-j}\right)$$
$$= f_r\left(p^k\right) + \sum_{j=0}^{k-1} f_r\left(p^j\right)\left\{\sum_{t=0}^{k-j} \mu\left(p^t\right)\left(\frac{u}{E_r}\right)\left(p^{k-j-t}\right)\right\}$$
$$= f_r\left(p^k\right) + \sum_{j=0}^{k-1} f_r\left(p^j\right)\left\{\left(\frac{u}{E_r}\right)\left(p^{k-j}\right) - \left(\frac{u}{E_r}\right)\left(p^{k-j-1}\right)\right\}$$
$$= f_r\left(p^k\right) + \sum_{j=0}^{k-1} f_r\left(p^j\right)\left\{\frac{1}{p^{(k-j)r}} - \frac{1}{p^{(k-j-1)r}}\right\}$$
$$= f_r\left(p^k\right) + \sum_{j=0}^{k-1} \frac{f_r\left(p^j\right)}{p^{(k-j-1)r}}\left(\frac{1}{p^r} - 1\right)$$
$$= f_r\left(p^k\right) - \left(1 - \frac{1}{p^r}\right)\sum_{j=0}^{k-1} \frac{f_r\left(p^j\right)}{p^{(k-j-1)r}}$$

(i) Now if $k = 1$, then (4.7) shows
$$\gamma_r(p) = f_r(p) - \left(1 - \frac{1}{p^r}\right)\frac{f_r(1)}{1}$$
$$= \frac{\varphi_r(p)}{\rho_r(p)} - \left(1 - \frac{1}{p^r}\right) = \frac{p^r - 1}{p^r} - \left(1 - \frac{1}{p^r}\right) = 0.$$

(ii) If $k = 2$, then (4.7) gives
$$\gamma_r\left(p^2\right) = f_r\left(p^2\right) - \left(1 - \frac{1}{p^r}\right)\left\{\frac{f_r(1)}{p^r} + \frac{f_r(p)}{1}\right\}$$
$$= \frac{\varphi_r\left(p^2\right)}{\rho_r\left(p^2\right)} - \left(1 - \frac{1}{p^r}\right)\left\{\frac{1}{p^r} + \frac{\varphi_r(p)}{\rho_r(p)}\right\}$$
$$= \frac{p^{2r} - p^r}{p^{2r} - p^r + 1} - \left(1 - \frac{1}{p^r}\right)\left(\frac{1}{p^r} + \frac{p^r - 1}{p^r}\right)$$
$$= \left(1 - \frac{1}{p^{2r} - p^r + 1}\right) - \left(1 - \frac{1}{p^r}\right)$$
$$= \frac{1}{p^r} - \frac{1}{p^{2r} - p^r + 1} < \frac{1}{p^r}.$$

(iii) Finally for $k \geq 3$, first note that

(4.8)
$$f_r\left(p^k\right)=\frac{p^{kr}-p^{(k-1)r}}{p^{kr}-p^{(k-1)r}+1}$$
$$=\frac{1-\frac{1}{p^r}}{1-\frac{1}{p^r}+\frac{1}{p^{kr}}}$$
$$=\left(1-\frac{1}{p^r}\right)\left\{1-\left(\frac{1}{p^r}-\frac{1}{p^{kr}}\right)\right\}^{-1}$$
$$=\left(1-\frac{1}{p^r}\right)\left\{1+\left(\frac{1}{p^r}-\frac{1}{p^{kr}}\right)+\left(\frac{1}{p^r}-\frac{1}{p^{kr}}\right)^2+\ldots\right\}$$
$$=\left(1-\frac{1}{p^r}\right)\left\{1+\left(\frac{1}{p^r}-\frac{1}{p^{kr}}\right)+\left(\frac{1}{p^{2r}}+\frac{1}{p^{2kr}}-\frac{2}{p^{(k+1)r}}\right)+\ldots\right\}$$
$$=\left(1-\frac{1}{p^r}\right)\left\{1+\frac{1}{p^r}+\frac{1}{p^{2r}}+\frac{1}{p^{2kr}}+O\left(\frac{1}{p^{(k+1)r}}\right)\right\}$$
$$=\left(1-\frac{1}{p^r}\right)+\left(\frac{1}{p^r}-\frac{1}{p^{2r}}\right)+\left(\frac{1}{p^{2r}}-\frac{1}{p^{3r}}\right)+\ldots$$
$$+\left(\frac{1}{p^{2kr}}-\frac{1}{p^{(k+1)r}}\right)+O\left(\frac{1}{p^{(k+1)r}}\right)$$
$$=1-\frac{1}{p^{kr}}+O\left(\frac{1}{p^{(k+1)r}}\right)$$

Now by (4.7) and (4.8) we have

(4.9)
$$\gamma_r\left(p^k\right)=f_r\left(p^k\right)-\left(1-\frac{1}{p^r}\right)\left\{f_r\left(p^{k-1}\right)+\frac{f_r\left(p^{k-2}\right)}{p^r}+\ldots+\frac{1}{p^{(k+1)r}}\right\}$$
$$=\left\{1-\frac{1}{p^{kr}}+O\left(\frac{1}{p^{(k+1)r}}\right)\right\}-\left(1-\frac{1}{p^r}\right)\left\{\left(1-\frac{1}{p^{(k-1)r}}+O\left(\frac{1}{p^{kr}}\right)\right)\right.$$
$$\left.+\frac{1}{p^{kr}}\left(1-\frac{1}{p^{(k-2)r}}+O\left(\frac{1}{p^{(k-1)r}}\right)\right)+\ldots+\frac{1}{p^{(k+1)r}}\right\}$$
$$=\left\{1-\frac{1}{p^{kr}}+O\left(\frac{1}{p^{(k+1)r}}\right)\right\}$$
$$-\left(1-\frac{1}{p^r}\right)\left\{1+\frac{1}{p^r}+\frac{1}{p^{2r}}+\ldots+\frac{1}{p^{(k-1)r}}-\frac{k-1}{p^{(k-1)r}}+O\left(\frac{1}{p^{(k+1)r}}\right)\right\}$$
$$=\left\{1-\frac{1}{p^{kr}}+O\left(\frac{1}{p^{(k+1)r}}\right)\right\}$$

$$-\left\{1+\frac{1}{p^r}+\frac{1}{p^{2r}}+\ldots+\frac{1}{p^{(k-1)r}}-\frac{k-1}{p^{(k-1)r}}+O\left(\frac{1}{p^{(k+1)r}}\right)\right\}$$
$$+\left\{\frac{1}{p^r}+\frac{1}{p^{2r}}+\ldots+\frac{1}{p^{kr}}-\frac{k-1}{p^{kr}}+O\left(\frac{1}{p^{(k+1)r}}\right)\right\}$$
$$=\frac{k-1}{p^{(k-1)r}}+O\left(\frac{1}{p^{rk}}\right)$$

since $r>1$ (by assumption) and $k\geq 3$ we have $(k-1)r\geq 2(k-1)>k+1$ so that for large value of p, say for $p>x_0$ we get from (4.9) that

$$\left|\gamma_r\left(p^k\right)\right|\leq\frac{1}{p^{k+1}},$$

proving part (iii) of the lemma.

4.10. Lemma. For $x\geq 1$,

(i) $$\sum_{n\leq x}\gamma_r(n)=O(1)$$

and (ii) $$\sum_{n\leq x}\frac{\gamma_r(n)}{n}=O\left(\frac{1}{x}\right)$$

Proof :- Note that $\gamma_r(n)=0$ if n is not a squarefull number by (i) of Lemma 4.6. Assume n is squarefull and such integer $n>1$ with $n=p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_t^{\alpha_t}$ can be written uniquely as $n=abc$, where a,b and c are pair wise co prime integers given by $a=\prod_{\substack{p_i>x_0\\ \alpha_i=2}}p_i^{\alpha_i}$, $b=\prod_{\substack{p_i>x_0\\ \alpha_i\geq 3}}p_i^{\alpha_i}$ and $c=\prod_{p_i\leq x_0}p_i^{\alpha_i}$ so that $\gamma_r(n)=\gamma_r(a)\gamma_r(b)\gamma_r(c)$.

Clearly $a=m^2$ where m is a squarefree integer. That is, $m=q_1^2q_2^2\ldots q_s^2$ where q_j's are distinct elements from $\{p_1,p_2,\ldots,p_t\}$. Therefore, by (ii) of Lemma 4.6,

$$|\gamma_r(a)|=\left|\gamma_r\left(q_1^2\right)\gamma_r\left(q_2^2\right)..\gamma_r\left(q_s^2\right)\right|\leq\frac{1}{q_1^r}.\frac{1}{q_2^r}\ldots\frac{1}{q_s^r}=\frac{1}{m^r}$$

so that

(4.11) $$\sum_{a\leq x}\gamma_r(a)\leq\sum_{m^2\leq x}\frac{|\mu(m)|}{m^r}=\sum_{m\leq\sqrt{x}}\frac{|\mu(m)|}{m^r}=O(1)$$

Also by (i) and (iii) of Lemma 4.6, we have

(4.12) $$\sum_{b\leq x}\gamma_r(b)\leq\prod_{p>x_0}\left\{1+\left|\gamma_r\left(p^3\right)\right|+\left|\gamma_r\left(p^4\right)\right|+\ldots\right\}$$
$$\leq\prod_{p>x_0}\left\{1+\frac{1}{p^4}+\frac{1}{p^5}+\ldots\right\}$$

$$\leq \prod_{p>x_0}\left\{1+\frac{1}{p^4}\frac{1}{\left(1-\frac{1}{p}\right)}\right\}$$
$$\leq \prod_{p>x_0}\left\{1+\frac{2}{p^4}\right\}$$
$$=O(1)$$

Finally, we get by the Lemma 4.6,

(4.13) $$\sum_{c\leq x}\gamma_r(c)\leq \prod_{p\leq x_0}\left\{1+\left|\gamma_r\left(p^2\right)\right|+\left|\gamma_r\left(p^3\right)\right|+\ldots\right\}$$
$$\leq \prod_{p\leq x_0}\left\{1+\frac{1}{p}+\frac{1}{p^3}+\frac{1}{p^4}+\ldots\right\}$$
$$\leq \prod_{p\leq x_0}\left\{1+\frac{1}{p}+\frac{2}{p^3}\right\}$$
$$=O(1)$$

Now combining (4.11), (4.12) and (4.13) we get

$$\sum_{n\leq x}\gamma_r(n)\leq \sum_{abc\leq x}\gamma_r(abc)$$
$$\leq \sum_{abc\leq x}\gamma_r(a)\gamma_r(b)\gamma_r(c)$$
$$\leq \left(\sum_{a\leq x}\gamma_r(a)\right)\left(\sum_{b\leq x}\gamma_r(b)\right)\left(\sum_{c\leq x}\gamma_r(c)\right)$$
$$=O(1),$$

proving the first part of the lemma.

Now by part (i) of the Lemma and Abel's identity imply

$$\sum_{n\leq x}\frac{\gamma_r(n)}{n}=\frac{1}{x}\sum_{n\leq x}\gamma_r(n)+\int_x^\infty\left(\sum_{t\leq x}\gamma_r(t)\right)\frac{1}{t^2}dt$$
$$=O\left(\frac{1}{x}\right)+O\left(\frac{1}{x}\right)=O\left(\frac{1}{x}\right),$$

proving (ii) of lemma.

4.14. Theorem. For $x\geq 1$,

$$\sum_{n\leq x}\frac{\varphi_r(n)}{\rho_r(n)}=C_r.x+O(\log x)$$

where $C_r=\frac{1}{\zeta(r+1)}\sum_{n=1}^{\infty}\frac{\gamma_r(n)}{n}$.

That is, the average order of $\frac{\varphi_r(n)}{\rho_r(n)}$ is C_r.

Proof :- By Lemma 4.2 and Lemma 4.10,

$$\sum_{n\le x}\frac{\varphi_r(n)}{\rho_r(n)}=\sum_{n\le x}f_r(n)$$
$$=\sum_{n\le x}\left(\frac{\varphi_r}{E_r}*\gamma_r\right)(n)$$
$$=\sum_{n\le x}\sum_{d\delta=n}\frac{\varphi_r(\delta)}{E_r(\delta)}.\gamma_r(d)$$
$$=\sum_{d\delta\le x}\gamma_r(d)\frac{\varphi_r(\delta)}{\delta^r}$$
$$=\sum_{d\le x}\gamma_r(d)\left\{\sum_{\delta\le\frac{x}{d}}\frac{\varphi_r(\delta)}{\delta^r}\right\}$$
$$=\sum_{d\le x}\gamma_r(d)\left\{\frac{\left(\frac{x}{d}\right)}{\zeta(r+1)}+O\left(\log\left(\frac{x}{d}\right)\right)\right\}$$
$$=\frac{x}{\zeta(r+1)}\sum_{d\le x}\frac{\gamma_r(d)}{d}++O\left(\log x\sum_{d\le x}\gamma_r(d)\right)$$
$$=x\left(C_r-\sum_{n>x}\frac{\gamma_r(n)}{n}\right)++O\left(\log x\sum_{n\le x}\gamma_r(n)\right)$$
$$=C_r.x+O\left(x\cdot\frac{1}{x}\right)+O(\log x)$$
$$=C_r.x+O(\log x),$$

proving the theorem.

4.15 Remark: The case $r=1$ of Theorem 4.14 was considered by László Tóth ([4], Theorem 5), who proved that

$$\sum_{n\le x}\frac{\rho(n)}{\varphi(n)}=C\,x+O\left((\log x)^{\frac{5}{3}}(\log\log x)^{\frac{4}{3}}\right),$$

where

$$C:=C_1=\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{\gamma_1(n)}{n}.$$

Observe that the error term in Theorem 4.14 is $O(\log x)$ and this is possible because we have considered the case of $r>1$ only.

5.REFERENCES:

[1].Apostol, Tom M., *Introduction to Analytic Number Theory*, Springer International Student Edition, Naroso Publishing House, New Delhi, 1998.
[2].Eckford Cohen, *Arithmetical Functions Associated with the Unitary Divisors of an Integer*, Math. Zeitschr. 74 (1960), 66-80.
[3].Klee, V. L, *Generalization of Euler's Function*, Amer. Math., Monthly, 55 (1948), 358-359.
[4].Laszlo Toth, *Regular Integers Modulo n*, Annals Univ. Sci. Budapest., Sect. Comp. 29 (2008), 263-275.
[5].McCarthy, Paul J., *Introduction to Arithmetical Functions*, Springer-Verlag, New York, 1985.
[6].Morgado, J, *Inteiors Regulares Modulo n*, Gazeta de Mathematical (Lisboa), 33 (1972), no.125-125, 1-5.
[7].Morgado, J. A *Property of the Euler φ-Function Concerning the Integers which are Regular Modulo n*, Portugal. Math., 33 (1974), 185-191.