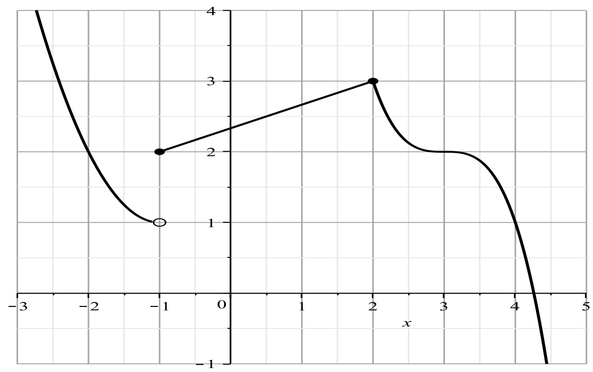
Math 1496 - Sample Test 1 Solutions

1. From the following graph determine the following limits.

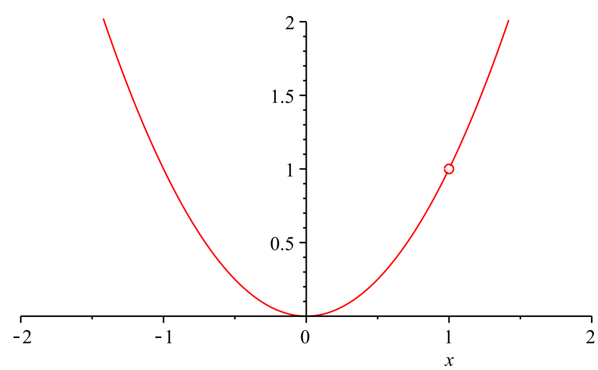


$(i)\quad \lim_{x\to -1^-} f(x) = 1 \qquad (ii)\quad \lim_{x\to -1^+} f(x) = 2 \qquad (iii)\quad \lim_{x\to -1} f(x) = \text{DNE}$

$(iv)\quad \lim_{x\to 2^-} f(x) = 3 \qquad (v)\quad \lim_{x\to 2^+} f(x) = 3 \qquad (vi)\quad \lim_{x\to 2} f(x) = 3$

2. Calculate $\lim_{x\to 1} \dfrac{x^3 - x^2}{x-1}$ using the techniques of graphically, numerically and analytically.

(i) Graphically



from which the graph says $\lim_{x\to 1} \dfrac{x^3 - x^2}{x-1} = 1.$

(ii) Numerically

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.8100	0.9801	0.9980	1.0020	1.0201	1.2100

from which the table says $\lim_{x\to 1} \dfrac{x^3 - x^2}{x-1} = 1.$

(iii) Analytically

$$\lim_{x\to 1}\frac{x^3-x^2}{x-1}=\lim_{x\to 1}\frac{x^2\cancel{(x-1)}}{\cancel{x-1}}=\lim_{x\to 1}x^2=1$$

3. Calculate the following limits analytically.

$$(i)\quad \lim_{x\to -1}\frac{x^2+3x+2}{x^2-1}=\lim_{x\to -1}\frac{(x+1)(x+2)}{(x+1)(x-1)}=\lim_{x\to -1}\frac{(x+2)}{(x-1)}=-\frac{1}{2}$$

$$(ii)\quad \lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}=\lim_{x\to 4}\frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}}=\lim_{x\to 4}\sqrt{x}+2=4$$

$$(iii)\quad \lim_{x\to 0}\frac{\sin 4x}{\sin 2x}=\lim_{x\to 0}\frac{\frac{\sin 4x}{x}}{\frac{\sin 2x}{x}}=\frac{4}{2}=2\quad\text{since}\quad\lim_{x\to 0}\frac{\sin mx}{x}=m$$

$$(iv)\quad \lim_{x\to\infty}\frac{3x^2+4}{x^2+2x+1}=\lim_{x\to\infty}\frac{3+\frac{4}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}}=\frac{3+0}{1+0+0}=3$$

4. Calculate the first derivative (either $f'(x)$ or y') of the following. Do not simplify your answer

$$(i)\quad y=\frac{4e^x}{x^2+1},\quad y'=\frac{4e^x(x^2+1)-4e^x\cdot 2x}{(x^2+1)^2}$$

$$(ii)\quad y=x^2\tan x,\quad y'=2x\tan x+x^2\sec^2 x$$

$$\begin{aligned}(iii)\quad y &= (2x+1)(x^2+3x+2),\\ y' &= 2(x^2+3x+2)+(2x+1)(2x+3)\\ &\ \text{at } x=-1, y'=(2)(0)+(-1)(1)=-1\end{aligned}$$

$$\begin{aligned}(iv)\quad y &= \frac{e^x}{x^2+1},\quad y'=\frac{e^x(x^2+1)-e^x2x}{(x^2+1)^2}\\ &\ \text{at } x=0, y'=\frac{e^0(1)-0}{(0+1)^2}=1\end{aligned}$$

5. The definition

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{n}\qquad(2)$$

If $f(x) = 3x^2 - 5x + 2$ then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{3(x+h)^2 - 5(x+h) + 2 - (3x^2 - 5x + 2)}{h} \\ &= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{5x} - 5h + \cancel{2} - \cancel{3x^2} + \cancel{5x} - \cancel{2}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(6x + 2h - 5)}{\cancel{h}} = \lim_{h\to 0} 6x + 2h - 5 = 6x - 5 \end{aligned}$$

6. If $y = x^4 - 2x^3 + 2x^2$ then $y' = 4x^3 - 6x^2 + 4x$.
At $x = 1, y = 1$ and $y'|_{x=1} = 4 - 6 + 4 = 2$
so the equation of the tangent is $y - 1 = 2(x - 1)$.

7. If

$$f(x) = \begin{cases} x + 1 & x \le 0 \\ x^2 + 1 & x > 0 \end{cases}$$

is $f(x)$ continuous and differentiable at $x = 0$?

Part (i)

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} x + 1 = 1$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} x^2 + 1 = 1$$

Further $f(0) = 0 + 1 = 1$. Since

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 1,$$

$f(x)$ is continuous at $x = 0$.

Part (ii)

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{x + 1 - 1}{x} = \lim_{x\to 0} 1 = 1$$
$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{x^2 + 1 - 1}{x} = \lim_{x\to 0} x = 0$$

Since

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x) = 0,$$

then $f(x)$ is not differentiable at $x = 0$.