Math 1497 - Calc 2

Techniques of Integration

1. u substitution

$\int \sin^2 x \cos x \, dx$ let $u = \sin x$
hard part $du = \cos x \, dx$

$\int u^2 du = \frac{u^3}{3} + c = \frac{1}{3}\sin^3 x + c$

2. parts

$\int u dv = uv - \int v du$

$\int x \sec^2 x \, dx$ $u = x$ $v = \tan x$
$du = dx$ $dv = \sec^2 x \, dx$

$= x \tan x - \int \tan x \, dx$

$= x \tan x + \ln|\cos x| + c$

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

$u = \cos x$

$du = -\sin x \, dx$

$-\int \frac{du}{u}$

3. Trig integrals

Form $\int \sin^m x \cos^n x\,dx$, $\int \tan^m x \sec^n x\,dx$

m, n integers

(i) m even, n odd
(ii) m odd, n even
(iii) both odd
(iv) both even

} we will look at these by example

Ex 1 $\int \sin^4 x \cos^3 x\,dx$

we will use a substitution
hard part is $(\sin x)^4$

$$\int \underbrace{\sin^4 x}_{u^4} \cos^2 x \underbrace{\cos x\,dx}_{du}$$

try $u = \sin x$
$du = \cos x\,dx$

split up
$\cos^3 x = \cos^2 x \cos x\,dx$

Now $\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x = 1 - u^2$

so our integral becomes

$$\int u^4(1-u^2)\,du = \int u^4 - u^6\,du = \frac{u^5}{5} - \frac{u^7}{7} + c$$

back sub $= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

<u>Ex 2</u> $\int \sin^3 x \cos^6 x\, dx$ $m=3$ odd $n=6$ even

↑ bad part

let $u = \cos x$ $du = -\sin x\, dx$ $\sin^2 x = 1-\cos^2 x$

$$\int \sin^2 x \cos^6 x \sin x\, dx$$

$$-\int (1-u^2)\, u^6\, du$$

$$-\int u^6 - u^8\, du = -\frac{u^7}{7} + \frac{u^9}{9} + c$$

$$= -\frac{1}{7}\cos^3 x + \frac{1}{9}\cos^9 x + c$$

Ex3 Both odd

$$\int \sin^3 x \cos^5 x \, dx$$

we could use $u = \sin x$ or $u = \cos x$

$u = \sin x \qquad du = \cos x \, dx \qquad \cos^4 x = (\cos^2 x)^2 = (1 - \sin^2 x)^2$

$$\int \sin^3 x \cos^4 x \cos x \, dx = \int u^3 (1-u^2)^2 \, du$$

$u = \cos x \qquad du = -\sin x \, dx$

$$\int \sin^2 x \cos^5 x \cos x \, dx \qquad \sin^2 x = 1 - \cos^2 x$$

$$-\int (1-u^2) u^5 \, du \leftarrow \text{easier}$$

$$-\int u^5 - u^7 \, du = -\frac{1}{5} u^5 + \frac{1}{7} u^7 + c$$

$$= -\frac{1}{5} \cos^5 x + \frac{1}{7} \cos^7 x + c$$

Ex4 m, n even

$\int \sin^2 x\,dx$ here we use trig identity

$\cos 2x = 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

so $\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

so $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}$

$\frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos 2x\,dx \leftarrow u = 2x \quad du = 2dx$

$\int \cos u \frac{du}{2} = \frac{\sin u}{2}$

$\frac{1}{2}x - \frac{1}{2}\cdot\frac{\sin 2x}{2} + c$

$= \frac{\sin 2x}{2}$

Ex5 $\int \sin^2 x \cos^2 x\,dx$

$\int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2}\,dx = \frac{1}{4}\int 1 - \cos^2 2x\,dx$

double angle formula again

we use $\cos^2\theta = \frac{1-\cos 2\theta}{2}$ replace $\theta = 2x$

so $\cos^2 2x = \frac{1-\cos 4x}{2}$

$$\frac{1}{4}\int 1\,dx - \frac{1}{4}\int \frac{1-\cos 4x}{2}\,dx$$

$$\frac{x}{4} - \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + c$$

$$= \frac{x}{8} + \frac{\sin 4x}{32} + c$$

Ex6 $\int_0^{\pi/4} \sin^5 x \cos^3 x \, dx$ $\quad u = \sin x \quad x = 0 \quad u = 0$

$\quad du = \cos x\, dx \quad x = \frac{\pi}{4} \quad u = \frac{\sqrt{2}}{2}$

$$\int_0^{\sqrt{2}/2} u^5 (1-u^2)\,du = \int_0^{\sqrt{2}/2} u^5 - u^7 \, du = \frac{u^6}{6} - \frac{u^8}{8}\Big|_0^{\sqrt{2}/2}$$

$$= \frac{\left(\frac{\sqrt{2}}{2}\right)^6}{6} - \frac{\left(\frac{\sqrt{2}}{2}\right)^8}{8} = \frac{1}{6}\cdot\frac{1}{8} - \frac{1}{8}\cdot\frac{1}{16} = \frac{5}{384}$$