International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 12 December, 2015 www.irjaf.com
DOI: 10.0612/article-2

Pricing of the Cross-Currency Interest Rate Guarantee Embedded in Financial Contracts in a LIBOR Market Model

Tsung-Yu Hsieh* Chi-Hsun Chou Son-Nan Chen

Abstract

We derive the pricing formulae for the financial contracts, such as guaranteed investment contracts (GICs), life insurance contracts, pension plans, and others, with the guaranteed minimum rate of return set relative to a LIBOR interest rate. Further, we analyze the guaranteed contracts in which the asset that provides the underlying return for the contract and the guaranteed interest rate are denominated in different currencies, which is a common practice. The guaranteed contracts with the above characteristics are called "cross-currency interest rate guaranteed contracts" (CIRGCs). Valuation of such contracts has not been investigated in the previous literature.

To value CIRGCs, a cross-currency LIBOR market model is introduced. The LIBOR market model for a single-currency economy is extended to a cross-currency economy which incorporates the traded-asset prices and exchange rate processes into the model setting. The cross-currency LIBOR market model (CLMM) is suitable and applicable to pricing a variety of CIRGCs. The pricing formulas derived under the CLMM are more tractable and feasible for practice than those derived under the instantaneous short rate model or the HJM model.

Four different types of CIRGCs are priced in this article. Calibration procedures are also discussed for practical implementation. In addition, Monte-Carlo simulation is provided to evaluate the accuracy of the theoretical prices.

Keywords： Interest rate, guarantee, Cross-currency, LIBOR market model
JEL: G12, G13, G22, G23

I. Introduction

Many real-word financial contracts have embedded some sort of minimum rate of return guarantee. Examples of such contracts could be guaranteed investment contracts (GICs), life insurance contracts, pension plans, and index-linked bonds. This leads to a tremendous amount of money managed by life insurance companies and pension funds. As a result, a further analysis of rate of return guarantees is warranted.

There are a variety of guarantee designs in financial contracts with guaranteed return in practice. One class of these guarantees is absolute guarantees, where the minimum rate of return is set to be deterministic. The other is the so-called relative guarantees in the literature (Lindset, 2004), where the minimum guaranteed rate of return is linked to a stochastic asset such as an index, a reference portfolio, an interest rate, a specific asset traded in financial markets, etc.

Previous research on valuing guarantees for life insurance products or pension funds has focused on absolute guarantees, which provide participants with a constant or predetermined minimum rate of return. The existing literature which analyzes absolute guarantees under the assumption of deterministic interest rate includes Brennan and Schwartz (1976), Boyle and Schwartz (1977), Boyle and Hardy (1997), and Grosen and Jorgensen (1997, 2000). Other researches conducted by adopting the Vasicek stochastic interest rate model (1977) include Persson and Aase (1997) and Hansen and Miltersen (2002). Miltersen and Persson (1999), Lindset (2003), and Bakken, Lindset and Olson (2006) adopt the Heath-Jarrow-Morton framework (HJM, 1992).

However, granting a deterministic guaranteed rate results in the inability to attract contract participants by a low guaranteed rate, while contract issuers bear financial burdens to attract contract participants with a high guaranteed rate. Consequently, a stochastic guaranteed rate, such as rate of return guarantees set relative to an interest rate or the rate of return on a mutual fund, has become more popular in recent developments. Despite the popularity of relative rate of return guarantees, especially those issued in Latin America, the relevant research is significantly less in number than absolute guarantees. Only a few articles were written on the relative rate of return guarantees. Ekern and Persson (1996) investigated unit-linked life insurance contracts with different types of relative guarantees. Pennacchi (1999) valued both the absolute and the relative guarantee provided for Chilean and Uruguayan defined contribution pension plans by employing a contingent claim analysis. Both papers assumed that interest rate was deterministic. However, Lindset (2004) analyzed a wide range of different kinds of minimum guaranteed rates of return within the HJM framework. The guaranteed rate of return examined in the above papers was set relative to the rates of return on equity-market assets. Moreover, Yang, Yueh and Tang (2008) extended their analysis to study rate of return guarantees relative to a return measured by market realized δ-year spot rates. The guarantees they examined were applied to all contributions in the accumulation period of a pension plan under the HJM model.

The guaranteed return contracts reflect the volatile nature of rates of return due to the fact that that market interest rates influence any rate of return process. A proper valuation model should consider the stochastic behavior of interest rates. However, the short rate models, such as the Vasicek model, the Cox, Ingersoll and Ross (CIR) model, and the HJM instantaneous

International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 12 December, 2015 www.irjaf.com
DOI: 10.0612/article-2

forward rate model have been extensively used for pricing contingent claims. Some problems should be noted for using the short rate modes or the HJM model.

First, the instantaneous short rate or the instantaneous forward rate is abstract and market-unobservable, and the underlying rate is continuously compounded, thus contradicting the market convention of being discretely compounded on the basis of the LIBOR rates. So the recovery of model parameters from market-observed data is a difficult and complicated task. Second, the pricing formulae of widely traded interest rate derivatives, such as caps, floors, swaptions, etc., based on the short rate models or the Gaussian HJM model are not consistent with market practice. This results in some difficulties in the parameter calibration procedure.

This research attempts to extend the previous analysis to value the financial contracts of the guaranteed minimum rate of return set relative to a LIBOR interest rate. Further, we analyze the guaranteed contracts in which the asset that provides the underlying return for the contract and the guaranteed interest rate are denominated in different currencies, which is a common practice. This common practice can always be observed in unit-linked products. The guaranteed contracts with the above characteristics are called "cross-currency interest rate guaranteed contracts" (CIRGCs, hereafter). However, all previous literature regarding guarantees assumes that both underlying assets and guaranteed rates in contracts are denominated in a single currency. This assumption is not consistent with the real economic environment and leads to pricing formulas unsuitable for valuing CIRGCs, since the "exchange-rate-effect" long discussed in the finance literature, is not fully reflected in pricing models. Amin and Jarrow (1991), Schlogl (2002), Musiela and Rutkowski (2005), and Wu and Chen (2007) show that the exchange-rate-effect affects the pricing results and should be considered in valuing cross-currency financial products. [1]

To value the CIRGCs, the popular LIBOR Market Model (LMM) is adopted in this research. Pricing CIRGCs under the LMM is more tractable for practice and avoids the problems shown in other interest rate models as mentioned earlier. Moreover, the "exchange-rate-effect" will be considered in this paper for pricing CIRGCs since the CIRGCs are linked to cross-currency assets. To achieve the goal, a cross-currency LIBOR Market Model (CLMM) is adopted to be the framework of the economic environment to derive the pricing formulae. The CLMM are more adequate and suitable for pricing CIRGCs. If the model setting degenerates to the single-currency case, the pricing CIRGCs model becomes the pricing model of the single-currency interest-rate guaranteed contract in the LMM framework.

Our article has several contributions to relative guarantee contracts. First, we use CLMM to derive the pricing formulae for the minimum return guaranteed contracts in which the guaranteed rate is set relative to the level of a stochastic LIBOR rate, which is different from the setting of the previous literatures based on continuous short rates or instantaneous forward rates. The interest rates used in the CLMM are consistent with conventional market quotes. As a result, all the model parameters can be easily obtained from market quotes, thus making the pricing formulae under the CLMM more tractable and feasible for practitioners.

Second, we analyze the cross-currency interest rate guarantee contracts which have not yet been studied in previous researches. The guaranteed contracts are often linked to cross-currency assets in practice. The interest rate guarantee embedded in cross-currency guaranteed contracts can be represented as an option which is equivalent to the quanto-type option in the finance literature. As a result, the exchange-rate-effect will appear in the pricing formulae of CIRGCs.

Third, the derived pricing formulae can be directly applied to pricing both maturity guarantees and multi-period interest guarantees with an arbitrary guarantee period. The pricing formula given by Yang, Yueh and Tang (2008) is available only for the guarantee period of one year. A maturity guarantee is binding only at the contract expiration. The cash flows connected to maturity guarantees are closely related to those of European options. For multi-period guarantees, the contract period is divided into several subperiods. A binding guarantee is specified for each subperiod. Many life insurance contracts and guaranteed investment contracts (GIC) sold by investment banks, cf. e.g., Walker (1992), are examples of multi-period guarantees. In addition, the derived pricing formulae of CIRGCs represent the general formulae for the interest rate guarantee under the CLMM. They can be applied to pricing the guarantees measured by the forward LIBOR rate and those measured by the spot rate which has been commonly used in the previous literature.

Fourth, using our pricing formulae is more efficient than adopting simulation, especially for those guaranteed contracts with long duration such as life insurance products and pension plans. The cross-currency interest rate guarantees embedded in contracts can be valued by recognizing their similarity to various Quanto types of “exotic” options. As a result, the pricing formulae of the CIRGCs within the CLMM framework can be derived via the martingale pricing method.

Fifth, we provide the calibration procedure for practical implementation and examine the accuracy of the pricing formulae via Monte-Carlo simulation.

The remainder of this paper is organized as follows. Section II briefly describes the setting of the economic model, i.e. the CLMM. In Section III, four different types of CIRGCs are defined and their pricing formulae under the framework of CLMM are derived. In Section IV, the calibration procedure for practical implementation is provided and the accuracy of the pricing formulae is examined via Monte-Carlo simulation. In Section V, the results of the paper are concluded with a brief summary.

II. Economic Model - Cross-Currency LIBOR Market Model

From the payoff structure of CIRGCs, CLMM can be adopted to be the setting of economic environment to develop the pricing formulas of CIRGCs. Based on the results of Amin and Jarrow (1991; AJ), the LMM has been extended to the CLMM by Wu and Chen (2007). Hence, we briefly specify the results of AJ in the first subsection. Next, the CLMM is introduced in the second subsection.[2] Different types of CIRGOs are priced under the CLMM framework in Section III.

II.1 The Results in AJ (1991)

Assume that trading takes place continuously in time over an interval $[0,\tau], 0<\tau<\infty$. The uncertainty is described by the filtered spot martingale probability space $\left(\Omega, F, Q, \{F_t\}_{t\in[0,\tau]}\right)$ where the filtration is generated by independent standard Brownian motions $W(t)=(W_1(t), W_2(t), ..., W_m(t))$. Q denotes the domestic spot martingale probability measure. The filtration $\{F_t\}_{t\in[0,\tau]}$ which satisfies the usual hypotheses represents the flow of information accruing to all the agents in the economy.[3] The notations are given below with *d* for domestic and *f* for foreign:

$f_k(t,T)$ = the k^{th} country's forward interest rate contracted at time *t* for instantaneous borrowing and lending at time *T* with $0\le t\le T\le\tau$, where $k\in\{d,f\}$.

$P_k(t,T)$ = the time *t* price of the k^{th} country's zero coupon bond (ZCB) paying one dollar at time *T*.

$S_k(t)$ = the time *t* price of the k^{th} country's asset (stock, index, or portfolio)

$r_k(t)$ = the k^{th} country's risk-free short rate at time *t*.

$\beta_k(t)$ = $\exp\left[\int_0^t r_k(u)\,du\right]$, the k^{th} country's money market account at time *t* with an initial value $\beta_k(0)=1$.

$X(t)$ = the spot exchange rate at $t\in[0,\tau]$ for one unit of foreign currency expressed in terms of domestic currency.

Based on the insights of Harrison and Kreps (1979), AJ (1991) extended the HJM model to a cross-currency case and clarified some conditions of the instantaneous forward rate process. Under these conditions, the market is arbitrage-free and complete and contingent claims can be priced by the risk-neutral valuation method. Their results are provided in the following proposition.

Proposition II.1 The Dynamics Under the Domestic Martingale Measure in AJ (1991)

For any $T \in [0,\tau]$, the dynamics of the forward rates, the ZCB prices, the asset prices and the exchange rate under the domestic martingale measure Q are given as follows:

$$df_d(t,T) = \sigma_{fd}(t,T)\cdot\sigma_{Pd}(t,T)dt + \sigma_{fd}(t,T)\cdot dW(t)$$

$$df_f(t,T) = \sigma_{ff}(t,T)\cdot\left[\sigma_{Pf}(t,T) - \sigma_X(t)\right]dt + \sigma_{ff}(t,T)\cdot dW(t)$$

$$\frac{dP_d(t,T)}{P_d(t,T)} = r_d(t)dt - \sigma_{Pd}(t,T)\cdot dW(t)$$

$$\frac{dP_f(t,T)}{P_f(t,T)} = \left[r_f(t) + \sigma_k(t)\cdot\sigma_{Pf}(t,T)\right]dt - \sigma_{Pf}(t,T)\cdot dW(t)$$

$$\frac{dS_d(t)}{S_d(t)} = r_d(t)dt - \sigma_{sd}(t,T)\cdot dW(t)$$

$$\frac{dS_f(t)}{S_f(t)} = \left[r_f(t) - \sigma_X(t)\cdot\sigma_{Sf}(t)\right]dt - \sigma_{Sf}(t)\cdot dW(t)$$

$$\frac{dX(t)}{X(t)} = \left[r_d(t) - r_f(t)\right]dt + \sigma_X(t)\cdot dW(t)$$

where $\sigma_{fk}(t,T)$ denotes the forward rate volatility of the domestic (k=d) or the foreign (k=f) country. Other double-subscript notations can be explained accordingly. The relationship between $\sigma_{fk}(t,T)$ and $\sigma_{Pk}(t,T)$ is given as follows:

$$\sigma_{Pk}(t,T) = \int_t^T \sigma_{fk}(t,u)\,du$$

The drift and volatility terms in Proposition 1 are subject to some regularity conditions.[4]

It is worth emphasizing that even in a cross-currency environment the drift term of the domestic forward rate under the domestic martingale measure Q still remains unchanged. However, for the foreign case, the drift has one additional term, $\sigma_{ff}(t,T)\cdot\sigma_X(t)$, which specifies the instantaneous correlation between the exchange rate and the foreign forward rate. It is also observed that the drift terms of the foreign assets are augmented by the instantaneous correlations between the exchange rate and the assets.

These arbitrage-free relationships between the volatility and the drift terms as given in Proposition II.1 can be employed to derive the arbitrage-free cross-currency LMM, which can be applied to pricing cross-currency interest rate guarantees.

II.2 The Cross-Currency LIBOR Market Model

The CLMM developed by Wu and Chen (2007) is briefly reported in this subsection. It is important to note that, thereafter, the term structure of interest rates is modeled by specifying the LIBOR rates dynamics, rather than the instantaneous forward rates dynamics. However, we still use the same notations and the same economic environment.

For some $\delta > 0,\ T \in [0,\tau]$ and $k \in \{d, f\}$, define the forward LIBOR rate process $\{L_k(t,T); 0 \le t \le T\}$ as given by

$$1 + \delta L_k(t,T) = \frac{P_k(t,T)}{P_k(t,T+\delta)} = \exp\left(\int_T^{T+\delta} f_k(t,u)\,du\right) \tag{2.1}$$

Assumption 1. A Family of Libor Rate Processes

Under the measure Q, $L_k(t,T)$, $k \in \{d, f\}$ *is assumed to have a lognormal volatility structure and its stochastic process is given by*

$$dL_k(t,T) = \mu_{Lk}(t,T)\,dt + L_k(t,T)\,\gamma_{Lk}(t,T)\cdot dW(t) \tag{2.2}$$

where $\gamma_{Lk}(\cdot,T):[0,T] \to \Re^m$ *is a deterministic, bounded, and piecewise continuous volatility function and* $\mu_{Lk}(t,T):[0,T] \to \Re$ *is some unspecified drift function.*

Assumption 2. The Asset Price Dynamics

Under the measure Q, $S_k(t,T)$, $k \in \{d, f\}$ *is assumed to have a lognormal volatility structure and its stochastic process is given by*

$$dS_k(t) = S_k(t)\,\mu_{Sk}(t)\,dt + S_k(t)\,\sigma_{Sk}(t)\cdot dW(t) \tag{2.3}$$

where $\sigma_{Sk}(t):[0,\tau] \to \Re^m$ *is a deterministic volatility vector function satisfying the standard regularity conditions and* $\mu_{Sk}(t)$ *is some drift function.*

Assumption 3. The Spot Exchange Rate Dynamics

Under the measure Q, the stochastic process of the spot exchange rate $X(t)$ *is given as follows:*

$$dX(t) = X(t)\,\mu_X(t)\,dt + X(t)\,\sigma_X(t)\cdot dW(t) \tag{2.4}$$

where $\mu_X(t):[0,\tau] \to \Re$ *is some unspecified drift function and* $\sigma_X(t):[0,\tau] \to \Re^m$ *is a deterministic process.*

For greater flexibility, the number of random shocks, *m*, is not precisely designated, but rather depends on the simplicity and accuracy required by the user. It is important to emphasize that the drift terms of the above stochastic processes are not yet determined. The specific forms of the drift terms must be chosen to make the economy arbitrage-free. The arbitrage-free relationship between the drift and the volatility terms in Proposition 2.1 is used by Wu and Chen (2007) to determine the drift terms in (2.2), (2.3), and (2.4) and given by:

$$\mu_{Sd}(t) = r_d(t),$$

$$\mu_{Sf}(t) = r_f(t) - \sigma_X(t)\cdot\sigma_{Sf}(t),$$

$$\mu_X(t) = r_d(t) - r_f(t).$$

The above results lead to the following Proposition.

Proposition II.2 The CLMM Under the Martingale Measure

Under the domestic spot martingale measure, the processes of the forward LIBOR rates and the exchange rate are expressed as follows:

$$\frac{dL_d(t,T)}{L_d(t,T)} = \gamma_{Ld}(t,T)\cdot\sigma_{Pd}(t,T+\delta)\,dt + \gamma_{Ld}(t,T)\cdot dW(t) \tag{2.5}$$

$$\frac{dL_f(t,T)}{L_f(t,T)} = \gamma_{Lf}(t,T)\cdot\left(\sigma_{Pf}(t,T+\delta) - \sigma_X(t)\right)dt + \gamma_{Lf}(t,T)\cdot dW(t) \tag{2.6}$$

$$\frac{dS_d(t)}{S_d(t)} = r_d(t)\,dt + \sigma_{Sd}(t)\cdot dW(t) \tag{2.7}$$

$$\frac{dS_f(t)}{S_f(t)} = \left[r_f(t) - \sigma_X(t)\cdot\sigma_{Sf}(t)\right]dt + \sigma_{Sf}(t,T)\cdot dW(t) \tag{2.8}$$

$$\frac{dX(t)}{X(t)} = \left(r_d(t) - r_f(t)\right)dt + \sigma_X(t)\cdot dW(t) \tag{2.9}$$

where $t \in [0,T],\ \ T \in [0,\tau]$ *and* $\sigma_{Pk}(t,T),\ \ k \in \{d,f\}$, *is defined below.*

$$\sigma_{Pk}(t,T) = \begin{cases} \displaystyle\sum_{j=1}^{[\delta^{-1}(T-t)]} \frac{\delta L_k(t,T-j\delta)}{1+\delta L_k(t,T-j\delta)}\gamma_{Lk}(t,T-j\delta) & t \in [0,T-\delta]\ ,\ T-\delta > 0,\ T \in [0,\tau] \\ 0 & \textit{otherwise.} \end{cases} \tag{2.10}$$

Unlike the abstract short rates in the instantaneous short rate models or the instantaneous forward rates in the HJM model, the forward LIBOR rates in the CLMM are market-observable. Furthermore, the cap pricing formula in the CLMM framework is consistent with the Black formula which is widely used in market practice and makes the calibration procedure easier. As a result, the volatility $\gamma_{Lk}(t,T),\ \ k \in \{d,f\}$, can be inverted from the interest rate derivatives traded in the market and $\sigma_{Pk}(t,T)$ and $k \in \{d,f\}$ can be calculated from equation (2.10).

According to the bond volatility process (2.10), $\{\sigma_{Pk}(t,T+\delta)\}_{t\in[0,T+\delta]}$ is stochastic rather than deterministic. To solve equation (2.5) and (2.6) for $L_k(T,T)$, Wu and Chen (2007) fix at initial time *s* and approximate $\sigma_{Pk}(t,T)$ by $\overline{\sigma}^{s}_{Pk}(t,T)$ given below:

$$\overline{\sigma}^{s}_{Pk}(t,T)=\begin{cases}\displaystyle\sum_{j=1}^{\left[\delta^{-1}(T-t)\right]}\frac{\delta L_k(s,T-j\delta)}{1+\delta L_k(s,T-j\delta)}\gamma_{Lk}(t,T-j\delta) & t\in[0,T-\delta]\;\&\;T-\delta>0,\\ 0 & otherwise.\end{cases}\tag{2.10}$$

where $0\le s\le t\le T\le\tau$. Hence, the calendar time of the process $\{F_k(t,T-j\delta)\}_{t\in[0,T-j\delta]}$ in (2.10) is frozen at its initial time s and the process $\{\overline{\sigma}^{s}_{Pk}(t,T)\}_{t\in[s,T]}$ becomes deterministic.

By substituting $\overline{\sigma}^{s}_{Pk}(t,T+\delta)$ for $\sigma_{Pk}(t,T+\delta)$ into the drift terms of (2.5) and (2.6), the drift and the volatility terms become deterministic, so we can solve (2.5) and (2.6) and find the approximate distribution of $L_k(T,T)$ to be lognormal.

The Wiener chaos order 0 approximation used in (2.10) is first utilized by BGM (1997) for pricing interest rate swaptions, developed further in Brace, Dun and Barton (1998), and formalized by Brace and Womersley (2000). It also appeared in Schlogl (2002). This approximation has been shown to be very accurate.

In Section III, four variants of the cross-currency interest-rate guaranteed contracts are priced based on the CLMM.

III. Valuation of Cross-Currency Interest Rate Guarantee Embedded in Financial Contracts

In this section, we define each type of financial contracts with cross-currency interest rate guarantees which are embedded in financial contracts as options. Then we derive the pricing formulae of four different types of cross-currency interest rate guarantees and the guaranteed contracts based on the cross-currency LIBOR market model. Introduction and analysis of each guarantee are presented sequentially as follows.

III.1 Valuation of First-Type Cross-Currency Interest Rate Guarantee

We define the guaranteed contracts first and then represent the interest rate guarantee as an option.

Definition III.1.1 *A financial contract with the payoff specified in (3.1.1) is called a*

First-Type Financial Contract with Cross-Currency Interest Rate Guarantee (FC_1CIRG)

$$FC_1(T+\delta) = N_d Max\left[\ S_f(T+\delta)/S_f(T)\ ,\ \ \left(1+\delta L_d^{\delta}(T,T)\right)\right] \quad (3.1.1)$$

where

N_d = *notional principal of the contract, in units of domestic currency*

$S_f(\eta)$ = *the underlying foreign asset price at time* η, $\eta \in [0,t,T,T+\delta]$

$L_d^{\delta}(T,T)$ = *the domestic T-matured LIBOR rates with a compounding period* δ

$P_d(t,\lambda)$ = *the time t price of the domestic zero coupon bond (ZCB) paying one dollar at time* λ, $\lambda \in \{T, T+\delta\}$.

T = *the start date of the guaranteed contract*

$T+\delta$ = *the expiry date of the guaranteed contract*

$(x)^+$ = $Max(x,0)$

$FC_1(T+\delta)$ can be rewritten as

$$FC_1(T+\delta) = N_d\left\{S_f(T+\delta)/S_f(T) + \left[\left(1+\delta L_d^{\delta}(T,T)\right) - S_f(T+\delta)/S_f(T)\right]^+\right\} \quad (3.1.2)$$

$$= N_d\left\{\left(1+\delta L_d^{\delta}(T,T)\right) + \left[S_f(T+\delta)/S_f(T) - \left(1+\delta L_d^{\delta}(T,T)\right)\right]^+\right\} \quad (3.1.3)$$

Equation (3.1.2) shows the payoff as the uncertain amount $S_f(T+\delta)/S_f(T)$ plus the maturity payoff of a put option written on the return of a reference foreign asset with a forward-start exercise price $\left(1+\delta L_d^{\delta}(T,T)\right)$. Alternatively, equation (3.1.3) indicates the payoff as the sum of the guaranteed amount $\left(1+\delta L_d^{\delta}(T,T)\right)$ and the final payoff of a call option to purchase the return of the reference foreign asset for the price $\left(1+\delta L_d^{\delta}(T,T)\right)$. Note that the exercise price, the guaranteed interest rate, is not decided at time *t* but is to be determined at future time *T*. For simplicity, the FC_1CIRG in (3.1.2) will be used for later analysis hereafter, which is employed in most guaranteed contracts in practice.

According to (3.1.2), we represent the interest rate guarantee embedded in the FC_1CIRG as an option below.

Definition III.1.2 *An option with the payoff specified in (3.1.4) is called a First-Type Cross-Currency Interest Rate Guarantee Option (C_1IRGO)*

$$C_1IRGO(T+\delta) = \left[\left(1+\delta L_d^{\delta}(T,T)\right) - S_f(T+\delta)/S_f(T)\right]^+, \quad (3.1.4)$$

There are several points worth noting. First, the guarantee of a minimum return, $\left(1+\delta L_d^{\delta}(T,T)\right)$, is set relative to the LIBOR rate, which is different from the setting of the previous literature that the interest rate is measured by continuous short rates or instantaneous forward rates. In addition, the LIBOR rate is quoted in markets. As a result, the CLMM is more appropriate for pricing *C_1IRGOs,* and all the parameters in the pricing formula can be easily obtained from market quotes, thus making the pricing formula more tractable and feasible for practitioners.

Second, we extend the analysis on the guaranteed contracts to the case where the underlying asset and the guaranteed interest rate are denominated in different currencies. An C_1IRGO is an option on the foreign-currency underlying asset $s_f(T+\delta)/s_f(T)$ with the domestic-currency exercise price $\left(1+\delta L_d^\delta(T,T)\right)$, and its final payments are denominated in domestic currency without directly incurring exchange rate risk. The previous researches on the minimum return guarantees use the assumption that both the underlying asset and the guaranteed interest rate are denominated in a single (domestic) currency. In practice, the guarantees (options) are often linked to a cross-currency asset. This is equivalent to a quanto-type option in the financial literature. As a result, the exchange-rate-effect will be reflected in the pricing model of CIRGOs which is more suitable for pricing cross-currency interest-rate guarantees, and if the model setting degenerates to the single-currency case, it reduces to the pricing model of interest rate guarantees in the LMM framework.

Third, the interest rate guarantee is set to begin at some future date T, rather than at current time t, and lasts for δ periods. The "forward-start" exercise price of this option, $1+\delta L_d^\delta(T,T)$, is unknown at date t and to be determined at some future date T. Hence, the guarantee for a minimum return over the period T to $T+\delta$ is analogous to a "forward-start" option. Setting the guarantee as the "forward-start" type has a notable advantage that the derived pricing formula of the maturity-guarantee can be directly applied to pricing the multi-period interest guarantees. However, only the pricing formulae for maturity-guarantee are presented for the parsimony sake.[5] In addition, the "forward-start" setting represents the general form of the interest guarantees. Specially, the guaranteed interest rate for the period t to $t+\delta$ ($T=t$) as measured by the spot rate in the previous literature can be obtained by setting $T=t$. As a result, the pricing formula of CIRGOs represents the general formulae for the interest rate guarantees under the CLMM and can be applied for pricing the guarantees measured by the spot LIBOR rates. Moreover, our formulae can be derived for arbitrary values of δ. In contrast, the formula of Yang, Yueh and Tang (2008) is available only for the special case where the interest rate guarantee is linked to the one-year spot rate, i.e. $\delta=1$, which will be examined later in Theorem 3.3.2. In addition, the "forward-start" pricing formulae provide more flexibility in the product design of interest-rate guarantees in practice.

Fourth, the cross-currency interest rate guarantees embedded in contracts can be valued by recognizing their similarity to various Quanto types of "exotic" options, such as "forward start options", "options to exchange one asset for another", and "options on the maximum of two risky assets". As a result, the pricing formulae of the CIRGCs within CLMM framework can be derived via the martingale pricing method.

The C_1IRGO pricing formula is expressed in the following theorem, and the proof is provided in Appendix A.

Theorem 3.1.1 *The pricing formula of a C_1IRGO with the final payoff as specified in (3.1.4) is expressed as follows:*

$$C_1IRGO(t)=N_dP_d(t,T+\delta)\left\{\left[1+\delta L_d^\delta(t,T)\right]N(-d_{12})-\left[1+\delta L_f^\delta(t,T)\right]e^{\int_t^T\theta_{11}(u)du+\int_T^{T+\delta}\theta_{12}(u)du}N(-d_{11})\right\}\quad(3.1.5)$$

where

$$d_{11}=\frac{\ln\left[\frac{1+\delta L_f^\delta(t,T)}{1+\delta L_d^\delta(t,T)}\right]+\int_t^T\theta_{11}(u)du+\int_T^{T+\delta}\theta_{12}(u)du+\frac{1}{2}V_1^2}{V_1}\ ,\qquad d_{12}=d_{11}-V_1$$

$$V_1^2 = \int_t^T \left(\left\|\theta_{13}(u)\right\|^2\right) du + \int_T^{T+\delta} \left\|\theta_{14}(u)\right\|^2 du$$

$$\theta_{11}(t) = \left[\sigma_X(t) - \overline{\sigma}_{P_f}(t,T+\delta) + \overline{\sigma}_{P_d}(t,T+\delta)\right] \cdot \left[\overline{\sigma}_{P_f}(t,T) - \overline{\sigma}_{P_f}(t,T+\delta)\right]$$

$$\theta_{12}(t) = \left[\sigma_X(t) - \overline{\sigma}_{P_f}(t,T+\delta) + \overline{\sigma}_{P_d}(t,T+\delta)\right] \cdot \left[-\overline{\sigma}_{P_f}(t,T+\delta) - \sigma_{S_f}(t)\right]$$

$$\theta_{13}(t) = \left[\overline{\sigma}_{P_f}(t,T+\delta) - \overline{\sigma}_{P_f}(t,T) + \overline{\sigma}_{P_d}(t,T) - \overline{\sigma}_{P_d}(t,T+\delta)\right]$$

$$\theta_{14}(t) = \sigma_{S_f}(t) + \overline{\sigma}_{P_f}(t,T+\delta)$$

$\overline{\sigma}^S_{Pk}(t,\cdot), k \in \{d,f\}$ is defined as (2.10).

The pricing equation (3.1.5) resembles the Margrabe (1978) type or the Black-type formula, but in the framework of the cross-currency LMM. Note that the terms, $\theta_{11}(t)$ and $\theta_{12}(t)$, appearing in (3.1.5) represent the effects of the exchange rate on pricing, which is induced by the fact that expected foreign cash flow is expressed under the domestic martingale measure and by the compound correlations between all the involved factors (the exchange rate and the domestic and foreign bonds).

Equation (3.1.5) can be used to price the market value of FC_1CIRGs at time t, and the pricing formula is given in the following theorem, and the proof is provided in Appendix A.

Theorem 3.1.2 *The time t market value of FC_1CIRGs with the final payoff as specified in (3.1.1) is given as follows:*

$$FC_1(t) = N_d P_d(t,T+\delta)\left\{\left[1+\delta L_d^\delta(t,T)\right]N(-d_{12}) + \left[1+\delta L_f^\delta(t,T)\right]e^{\int_t^T \theta_{11}(u)du + \int_T^{T+\delta}\theta_{12}(u)du} N(d_{11})\right\} \quad (3.1.6)$$

Note that the advantage of adopting the cross-currency BGM model rather than other interest-rate models is that all the parameters in (3.1.5) and (3.1.6) can be easily obtained from market quotes, thus making the pricing formula more tractable and feasible for practitioners.

III.2 Valuation of Second-Type Cross-Currency Interest Rate Guarantee

Definition 3.2.1 *A financial contract with the payoff specified in (3.2.1) is called a Second-Type Financial Contract with Cross-Currency Interest Rate Guarantee (FC_2CIRG)*

$$FC_2(T+\delta) = N_d Max\left[S_d(T+\delta)/S_d(T) \quad , \quad \left(1+\delta L_f^\delta(T,T)\right)\right] \quad (3.2.1)$$

$S_d(\eta)$ = *the underlying domestic asset price at time* η, $\eta \in [0,t,T,T+\delta]$

$L_f^\delta(T,T)$ = *the foreign T-matured LIBOR rate with a compounding period* δ

Similar to FC_1CIRG, the maturity payoff of FC_2CIRG can be rewritten as follows.

$$FC_2(T+\delta) = N_d\left\{S_d(T+\delta)/S_d(T) + \left[\left(1+\delta L_f^\delta(T,T)\right) - S_d(T+\delta)/S_d(T)\right]^+\right\} \quad (3.2.2)$$

Definition 3.2.2 *An option with the payoff specified in (3.2.3) is called a Second-Type Cross-Currency Interest Rate Guarantee Option (C_2IRGO),*

$$C_2IRGO(T+\delta) = \left[\left(1+\delta L_f^\delta(T,T)\right) - S_d(T+\delta)/S_d(T)\right]^+ \quad (3.2.3)$$

The difference between C_2IRGO and C_1IRGO is that C_2IRGO is written on the domestic underlying asset, $S_d(T+\delta)/S_d(T)$, with the foreign exercise price, $1+\delta L_f^\delta(T,T)$. *C_2IRGO* bears much resemblance to *C_1IRGO* as mentioned in the previous section.

Next, we begin with pricing the *C_2IRGO*. The resulting formula of the *C_2IRGO* is then used to value *FC_2CIRGs*. The C_2IRGOs pricing formula is given in Theorem 3.2.1 below and the proof is provided in Appendix B.

Theorem 3.2.1 *The pricing formula of C_2IRGOs with the final payoff as specified in (3.2.3) is presented as follows:*

$$C_2IRGO(t) = N_d P_d(t,T+\delta)\left\{\left[1+\delta L_f^\delta(t,T)\right]e^{-\int_t^T \theta_{21}(u)du} N(-d_{22}) - \left[1+\delta L_d^\delta(t,T)\right]N(-d_{21})\right\} \tag{3.2.4}$$

$$d_{21} = \frac{\ln\left[\dfrac{1+\delta L_d^\delta(t,T)}{1+\delta L_f^\delta(t,T)}\right] + \int_t^T \theta_{21}(u)du + \frac{1}{2}V_2^2}{V_2}, \quad d_{22} = d_{21} - V_2$$

$$V_2^2 = \int_t^T \left(\left\|\theta_{22}(u)\right\|^2\right)du + \int_T^{T+\delta}\left\|\theta_{23}(u)\right\|^2 du$$

$$\theta_{21}(t) = \left[\sigma_X(t) - \overline{\sigma}_{P_f}(t,T+\delta) + \overline{\sigma}_{P_d}(t,T+\delta)\right]\cdot\left[\overline{\sigma}_{P_f}(t,T+\delta) - \overline{\sigma}_{P_f}(t,T)\right]$$

$$\theta_{22}(t) = \left[\overline{\sigma}_{P_d}(t,T+\delta) - \overline{\sigma}_{P_d}(t,T) + \overline{\sigma}_{P_f}(t,T) - \overline{\sigma}_{P_f}(t,T+\delta)\right]$$

$$\theta_{23}(t) = \left[\sigma_{S_d}(t) + \overline{\sigma}_{P_d}(t,T+\delta)\right]$$

Similar to C_1IRGOs, the effect of the exchange rate $\theta_{21}(t)$ still appears in (3.2.4), although the maturity payoff is denominated in domestic currency without directly incurring exchange rate risk. Note that the influence of the exchange rate on C_2IRGOs lasts only from period *t* to *T* while it lasts from *t* to *T*+δ on C_1IRGOs. In addition, the foreign-currency denominated exercise price, $\left(1+\delta L_f^\delta(T,T)\right)$, in C_2IRGOs is stochastic from period t to T, but known over the period from *T* to *T*+δ. In contrast, the counterpart asset, $S_f(T+\delta)/S_f(T)$, in C_1IRGOs is stochastic from period *t* to *T*+δ, and hence the exchange rate impact on the C_1IRGOs pricing extends further to the period from *T* to *T*+δ.

Equation (3.2.4) is utilized to price *FC_2CIRGs* at time *t*, and the pricing formula is given in the following theorem. The proof is provided in Appendix B.

Theorem 3.2.2 *The market value at time t of FC_2CIRGs with the final payoff as specified in (3.2.1) is expressed as follows:*

$$FC_2(t) = N_d P_d(t,T+\delta)\left\{\left[1+\delta L_f^\delta(t,T)\right]e^{\int_t^T \theta_{21}(u)du} N(-d_{22}) + \left[1+\delta L_d^\delta(t,T)\right]N(d_{21})\right\} \tag{3.2.5}$$

III.3 Valuation of Third-Type Cross-Currency Interest Rate Guarantee

Definition 3.3.1 *A financial contract with the payoff specified in (3.3.1) is called a Third-Type Financial Contract with Cross-Currency Interest Rate Guarantee (FC_3CIRG)*

$$FC_3(T+\delta) = N_d Max\left[S_f(T+\delta)/S_f(T)\ ,\ \left(1+\delta L_f^\delta(T,T)\right)\right] \tag{3.3.1}$$

Once again, the expiry payoff of the *FC_3CIRG* is rewritten as follows:

$$FC_3(T+\delta) = N_d\left\{S_f(T+\delta)/S_f(T) + \left[\left(1+\delta L_f^\delta(T,T)\right) - S_f(T+\delta)/S_f(T)\right]^+\right\} \tag{3.3.2}$$

Based on (3.3.2), we define the option embedded in the *FC_3CIRG* below.

Definition 3.3.2 *An option with the payoff specified in (3.3.3) is called a Third-Type Cross-Currency Interest Rate Guarantee Option (C_3IRGO)*

$$C_3IRGO(T+\delta)=N_d\left[\left(1+\delta L_f^{\delta}(T,T)\right)-S_f(T+\delta)/S_f(T)\right]^+, \tag{3.3.3}$$

Different from C_1IRGOs and C_2IRGOs, an C_3IRGO is an option written on the difference between the return on the foreign underlying asset, $S_f(T+\delta)/S_f(T)$, and the foreign interest rate, $1+\delta L_f^{\delta}(T,T)$, for period t to T+δ, but the final payment is measured in domestic currency. The holders of this guaranteed contract also have the advantage of avoiding direct exchange rate risk.

Since the C_3IRGO can be priced via the martingale method under the CLMM, we omit the proof to keep the reasonable length of our paper.[6]

Theorem 3.3.1 *The pricing formula of C_3IRGOs with the final payoff as specified in (3.3.3) is presented as follows:*

$$C_3IRGO(t)$$
$$=N_dP_d(t,T+\delta)\left\{\left[1+\delta L_f^{\delta}(t,T)\right]e^{\int_t^T\theta_{31}(u)du}N(-d_{32})-\left[1+\delta L_f^{\delta}(t,T)\right]e^{\int_t^T\theta_{31}(u)du+\int_T^{T+\delta}\theta_{32}(u)du}N(-d_{31})\right\} \tag{3.3.4}$$

where

$$d_{31}=\frac{\int_T^{T+\delta}\theta_{32}(u)\,du+\frac{1}{2}V_3^2}{V_3}\quad,\quad d_{32}=d_{31}-V_3$$

$$V_3^2=\int_T^{T+\delta}\left\|\theta_{33}(u)\right\|^2du$$

$$\theta_{32}(t)=\left[\sigma_X(t)-\bar{\sigma}_{P_f}(t,T+\delta)+\bar{\sigma}_{P_d}(t,T+\delta)\right]\cdot\left[-\bar{\sigma}_{P_f}(t,T+\delta)-\sigma_{S_f}(t)\right]$$

$$\theta_{31}(t)=\left[\sigma_X(t)-\bar{\sigma}_{P_f}(t,T+\delta)+\bar{\sigma}_{P_d}(t,T+\delta)\right]\cdot\left[\bar{\sigma}_{P_f}(t,T)-\bar{\sigma}_{P_f}(t,T+\delta)\right]$$

$$\theta_{33}(t)=\sigma_{S_f}(t)+\bar{\sigma}_{P_f}(t,T+\delta)$$

Similarly, although the maturity payoff is measured in domestic currency without directly involving the exchange rate, the exchange rate impact still is presented in (3.3.4) as shown by $\theta_{31}(t)$ and $\theta_{32}(t)$. Note that the exchange rate impact on C_3IRGOs lasts for the whole period from t to T+δ. The foreign-currency denominated interest rate, $\left(1+\delta L_f^{\delta}(T,T)\right)$, in C_3IGOs is stochastic from period t to T, but known over the period from T to T+δ, while the stochastic nature of the foreign-currency denominated asset, $S_f(T+\delta)/S_f(T)$, prevails over the whole period from t to T+δ. As a result, the exchange rate affects the C_3IGOs pricing in a different way over the intervals [t, T] and [T, T+δ].

Once again, (3.3.4) is used to price *FC_3IRGs* at time t, and the pricing formula is given in the following theorem.

Theorem 3.3.2 *The market value at time t of FC_3IRGs with the final payoff as specified in (3.3.1) is expressed as follows:*

DOI: 10.0612/article-2

$$FC_3(t) = N_d P_d(t, T+\delta)\left\{\left[1+\delta L_f^{\delta}(t,T)\right] e^{\int_t^T \theta_{33}(u)du} N(-d_{32}) + \left[1+\delta L_f^{\delta}(t,T)\right] e^{\int_t^T \theta_{31}(u)du + \int_T^{T+\delta} \theta_{32}(u)du} N(d_{31})\right\} \quad (3.3.5)$$

Yang, Yueh and Tang (2008) have derived under the HJM framework the pricing formulae for interest rate guarantee options, which are written on the underlying difference between the return on a domestic asset and a domestic interest rate, denominated in domestic currency. However, their pricing formula can not be used for pricing the options which are linked to the cross-currency assets. In comparison with their pricing formula, the major difference between Theorem 3.3.2 and their formula lies in the fact that not only the "exchange-rate-effect" is considered in Theorem 3.3.2, but also all the parameters in Theorem 3.3.2 can be extracted from market quotes, which makes our pricing formula more tractable and feasible for practitioners. Besides, their setting of the guaranteed interest rate measured by the spot rate is a special case of our types. Moreover, our formula can be derived for arbitrary values of δ. In addition, their formula derived under the HJM framework is available only for the special case where the interest rate guarantee is linked to the one-year spot rate, i.e. δ=1, in the pricing of multi-period rate of return guarantee.

3.4 Valuation of Fourth-Type Cross-Currency Interest Rate Guarantee

Definition 3.4.1 *A financial contract with the payoff specified in (3.4.1) is called a Fourth-Type Financial Contract with Interest Rate Guarantee (FC_4CIRG)*

$$FC_4(T+\delta) = X(T+\delta)\, N_f Max\left[S_f(T+\delta)/S_f(T) \ , \ \left(1+\delta L_f^{\delta}(T,T)\right)\right] \quad (3.4.1)$$

$X(T+\delta)$ = *the floating exchange rate at time $T+\delta$ expressed as the domestic currency value of one unit of foreign currency.*

N_f = *notional principal of the contract, in units of foreign currency.*

The expiry payoff of *FC_4CIRGs* can be expressed as follows.

$$FC_4(T+\delta) = X(T+\delta)\, N_f \left\{ S_f(T+\delta)/S_f(T) + \left[\left(1+\delta L_f^{\delta}(T,T)\right) - S_f(T+\delta)/S_f(T)\right]^{+}\right\} \quad (3.4.2)$$

We define the option embedded in this contract below.

Definition 3.4.2 *An option with the payoff specified in (3.4.2) is called a Fourth-Type Cross-Currency Interest Rate Guarantee Option (C_4IRGO)*

$$C_4IRGO(T+\delta) = X(T+\delta)\, N_f \left[\left(1+\delta L_f^{\delta}(T,T)\right) - S_f(T+\delta)/S_f(T)\right]^{+}, \quad (3.4.3)$$

From the viewpoint of domestic investors, holding an C_4IRGO acts much in the same way as longing an option, whose payoff is based on the difference between the foreign interest rate and the return on the underlying foreign asset, both denominated in foreign currency. The foreign-currency payoff is converted via multiplying the floating exchange rate into the domestic-currency payoff. The structure of an C_4IRGO is different from that of an C_3IRGO in that this option is directly affected by movements in the exchange rate. If the exchange rate moves upward, a holder of this option may enhance profits from the exchange rate gain when the option is in the money at expiry.

International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 12 December, 2015 www.irjaf.com
DOI: 10.0612/article-2

Since the C_4IRGO can be also priced via the martingale method under the CLMM as the C_3IRGO, we omit the proof.[7] The pricing formula of C_4IRGOs is given below.

Theorem 3.4.1 *The pricing formula of C_4IRGOs with the final payoff as specified in (3.4.2) is presented as follows:*

$$C_4IRGO(t) = X(t) N_f P_f(t,T+\delta)\left\{\left[1+\delta L_f^{\delta}(t,T)\right]N(-d_{42}) - \left[1+\delta L_f^{\delta}(t,T)\right]N(-d_{41})\right\} \quad (3.4.4)$$

$$d_{41} = \frac{1}{2}V \ , \quad d_{42} = d_{41} - V_4$$

$$V_4^2 = \int_T^{T+\delta} \left\|\theta_4(u)\right\|^2 du$$

$$\theta_4(t) = \left[\sigma_{S_f}(t) + \bar{\sigma}_{P_f}(t,T+\delta)\right]$$

By observing (3.4.3), the option pricing formula is directly affected by unanticipated changes in the exchange rate since the expiry payoff is determined by the spot exchange rate at time $T+\delta$. The pricing formula shows that the option can be first priced under the foreign forward martingale measure and then the foreign-currency fair price is converted via multiplying the time t spot exchange rate $X(t)$ into the domestic-currency market fair value.

Equation (3.4.4) is used to price *FC_4IRGs,* and the pricing formula is represented in the following theorem.

Theorem 3.4.2 *The market value at time t of FC_4IRGs with the final payoff as specified in (3.4.1) is expressed as follows:*

$$FC_4(t) = N_f X(t) P_f(t,T+\delta)\left\{\left[1+\delta L_f^{\delta}(t,T)\right]N(-d_{42}) + \left[1+\delta L_f^{\delta}(t,T)\right]N(d_{41})\right\} \quad (3.4.5)$$

The above four different pricing formulae of cross-currency interest rate guarantees have been derived. In section 4, we are devoted to some practical issues regarding a calibration procedure and numerical examples.

IV. Calibration Procedure and Numerical Examples

In this section, we first provide a calibration procedure for practical implementation and then examine the accuracy of the derived pricing formulae via a comparison with Monte Carlo simulation.

IV.1 Calibration Procedure

In this subsection, the mechanism presented by Rebonato (1999) and employed by Wu and Chen (2007) is introduced to engage in a simultaneous calibration of the extended LMM to the percentage volatilities and the correlation matrix of the underlying forward LIBOR rates, the exchange rate, and the domestic and foreign equity assets (which are assumed to be stock indexes hereafter and could be stocks, mutual funds, or reference portfolios). We briefly report it below.[8]

To illustrate the procedures, we assume that there are n domestic forward LIBOR rates, n foreign forward LIBOR rates, a domestic stock index, a foreign stock index, and an exchange rate in an m-factor framework. The steps to calibrate the model parameters are briefly presented below:

First, the instantaneous volatility of the forward LIBOR rate $\gamma_k(t,T_i), k \in \{d,f\}$ is chosen

International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 12 December, 2015 www.irjaf.com
DOI: 10.0612/article-2

such that the caplet price is correctly recovered. In other words, the market-quoted volatility for T_i years from the current time t caplet, denoted by $\sigma_{k,imp}(T_i - t)$, is given by

$$\int_t^{T_i} \left\| \gamma_k(u, T_i) \right\|^2 du = \sigma_{k,imp}^2 (T_i - t) \times (T_i - t).$$

Next, we assume that the current market term structure is recovered by a piecewise-constant instantaneous total volatility depending on only the duration of the underlying rates. The time-dependent volatility of each period for each rate can be determined by the stripping method presented in Rebonato (1999). To describe the volatility structure engaged in the extended LMM, the element $S_{k,i-j}$ is defined by

$$\sum_{j=1}^{i} S_{k,i-j}^2 (T_j - T_{j-1}) = \sigma_{k,imp}^2 (T_i - t) \times (T_i - t), \quad i = 1, 2, \ldots, n,$$

where the subscript i labels the ith domestic/foreign forward LIBOR rate, thus converting the instantaneous volatility into a time-dependent volatility structure. The element $S_{k,i-j}$ is solved by the recurrence relation shown below:

$$S_k \equiv \begin{pmatrix} S_{k,0} & & & & \\ S_{k,1} & S_{k,0} & & & \\ S_{k,2} & S_{k,1} & S_{k,0} & & \\ \vdots & \vdots & \vdots & \ddots & \\ S_{k,n-1} & S_{k,N-2} & S_{k,N-3} & \cdots & S_{k,0} \end{pmatrix}.$$

Moreover, the instantaneous volatility of the exchange rate and the foreign stock index can be calculated directly from the historical data of the underlying exchange rate and the foreign index in the market or extracted from the on-the-run options prices in practice. To ease analysis, the term structure of volatility is assumed to be flat and calculated from historical data.

Second, the correlation structure of the domestic and foreign forward LIBOR rates, the domestic and foreign stock index, and the exchange rate is given by

$$dW(t) dW'(t) = \sum dt,$$

where $\sum$ is the market correlation matrix of the domestic and foreign forward LIBOR rates, the domestic and foreign stock index, and the exchange rate. $\sum$ is an $(2n+3)$-rank, positive-definite, and symmetric matrix and can be written as follows:

$$\sum = AA' = P\Phi P',$$

where P is a real orthogonal matrix and Φ is a diagonal matrix. A is defined by $A \equiv P\Phi^{1/2}$. Next, the full-rank correlation matrix $\sum$ is approximated by selecting a suitable correlation matrix $\sum^B$ that can be decomposed by a m-rank matrix B with $m \leq 2n+3$, where $dZ(t)$ is a vector of m-dimensional Brownian motions:

$$BdZ(t)\left(BdZ(t)\right)' = BB'dt = \sum^B dt.$$

The appropriate choice of the matrix B is as proposed by Rebonato (1999):

$$b_{i,k} = \begin{cases} \cos\theta_{i,k} \Pi_{j=1}^{k-1} \sin\theta_{i,j} & if \ k = 1, 2, \ldots, m-1 \\ \Pi_{j=1}^{k-1} \sin\theta_{i,j} & if \ k = m \end{cases}.$$

Then, a suitable matrix $\hat{B}$ is obtained by finding the solution of $\hat{\theta}$ for the optimization

International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 12 December, 2015 www.irjaf.com
DOI: 10.0612/article-2

problem:

$$\min_{\theta} \sum_{i,j=1}^{2n+3} \left| \Gamma_{i,j}^{B} - \Gamma_{i,j} \right|^2 .$$

Hence, the approximated correlation matrix $\sum^B = \hat{B}\hat{B}'$ mimics the correlation of the market $\sum$.[9]

Via the distributing matrix B, the individual instantaneous volatility applied to each Brownian motion at each period for each process can be derived. The data calibrated from the market correlation matrix and volatilities can be used to calculate the prices of the CIRGOs and the guaranteed contracts as derived in Theorem 3.1.1 to 3.4.2.

IV.2 Numerical Analysis

Some practical examples are given to examine the accuracy of the pricing formulae derived in the previous section and compare the results with Monte Carlo simulation. Based on actual 2-year market data,[10] four types of FCIRGs with different guarantee periods are priced, and the results are listed in Exhibit 3 and 4. The notional value is assumed to be \$1. The simulation is based on 50,000 sample paths. The domestic country is the U.S. and the foreign country is the U.K in the examples. The domestic stock index is the Dow Jones Industrials and the foreign index is the FTSE index.

Exhibit 1 and 2 show the prices of four types of FCIRGs with δ=1 and δ=0.5. Observing the numerical results yields several notable points. First, the pricing formulae have been shown to be accurate and robust in comparison with Monte Carlo simulation for the recent market data. Second, Exhibit 2 shows that our formulae can be applied for arbitrary values of δ (other than δ=1). The formula of Yang, Yueh and Tang (2008) is available only for the special case where the interest rate guarantee is linked to the one-year spot rate, i.e. δ=1. Third, the value of FCIRGs decreases with the longer start date T for each type of FCIRGs with a fixed guarantee period δ. Fourth, the value of FC_4IRGs is higher than those of the other three FCIRGs since FC_4IRG is directly affected by the spot exchange rate. Finally, using the derived formulae is more efficient than adopting simulation for those guaranteed contracts with long duration such as life insurance products and pension plans.

Exhibit 1. The Prices of Four Types of FCCIRGs with δ=1 Year

	FC_1CIRG			*FC_2CIRG*		
$(t,T,T+\delta)$	FC	MC	SE	FC	MC	SE
(0,1,2)	104.9482%	105.0205%	0.0573%	105.5409%	105.4854%	0.0518%
(0,2,3)	100.5584%	100.5033%	0.0530%	101.1014%	101.1056%	0.0516%
(0,3,4)	96.1686%	96.1535%	0.0507%	96.6757%	96.6476%	0.0502%
(0,4,5)	92.0138%	91.9323%	0.0476%	92.4900%	92.5183%	0.0490%
(0,5,6)	88.1651%	88.2050%	0.0458%	88.6139%	88.5980%	0.0473%
(0,10,11)	72.4778%	72.4630%	0.0366%	72.8369%	72.8375%	0.0392%
(0,15,16)	61.3509%	61.3433%	0.0309%	61.6537%	61.6381%	0.0333%
(0,20,21)	53.2200%	53.2309%	0.0270%	53.4808%	53.4672%	0.0288%
(0,25,26)	47.0944%	47.1129%	0.0234%	47.3252%	47.3243%	0.0255%
(0,30,31)	42.3865%	42.3714%	0.0212%	42.5952%	42.5597%	0.0228%

International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 12 December, 2015 www.irjaf.com
DOI: 10.0612/article-2

	FC₃CIRG			*FC₄CIRG*		
($t,T,T+\delta$)	FC	MC	SE	FC	MC	SE
(0,1,2)	106.0262%	105.9490%	0.0539%	201.8597%	201.6028%	0.1859%
(0,2,3)	101.0910%	101.1032%	0.0521%	190.2855%	190.4313%	0.1873%
(0,3,4)	96.4282%	96.4194%	0.0497%	180.2621%	180.2126%	0.1825%
(0,4,5)	92.0806%	92.0704%	0.0473%	171.5524%	171.4857%	0.1785%
(0,5,6)	88.0760%	88.0850%	0.0456%	164.0535%	164.0666%	0.1743%
(0,10,11)	72.2113%	72.2085%	0.0372%	136.8541%	136.7752%	0.1616%
(0,15,16)	61.1070%	61.1089%	0.0314%	118.9955%	118.8971%	0.1583%
(0,20,21)	52.9729%	52.9796%	0.0271%	106.6352%	106.7059%	0.1616%
(0,25,26)	46.8762%	46.8679%	0.0239%	97.8470%	97.7910%	0.1686%
(0,30,31)	42.2142%	42.2089%	0.0217%	91.1130%	91.0398%	0.1795%

The abbreviations FC, MC and SE represent the results of the formula, Monte Carlo simulations, and the standard error, respectively. The current time, the start date, and the expiry date of the guaranteed contract are represented by t, T and T+δ, respectively.

Exhibit 2. The Prices of Four Types of FCCIRGs with δ=0.5 Year

	FC_1CIRG			FC_2CIRG		
($t,T,T+\delta$)	FC	MC	SE	FC	MC	SE
(0,1,1.5)	101.7899%	101.7872%	0.0359%	102.0085%	102.0480%	0.0383%
(0,2,2.5)	97.9812%	97.8913%	0.0343%	98.1916%	98.1481%	0.0361%
(0,3,3.5)	93.9803%	93.9930%	0.0334%	94.1842%	94.1955%	0.0352%
(0,4,4.5)	90.0914%	90.1058%	0.0322%	90.2872%	90.2243%	0.0332%
(0,5,5.5)	86.4618%	86.4668%	0.0310%	86.6498%	86.6385%	0.0321%
(0,10,10.5)	71.2645%	71.2810%	0.0255%	71.4198%	71.4003%	0.0264%
(0,15,15.5)	60.3258%	60.3029%	0.0213%	60.4563%	60.4680%	0.0225%
(0,20,20.5)	52.2998%	52.3037%	0.0186%	52.4107%	52.4330%	0.0198%
(0,25,25.5)	46.2952%	46.2872%	0.0163%	46.3939%	46.4037%	0.0174%
(0,30,30.5)	41.6685%	41.6493%	0.0146%	41.7580%	41.7679%	0.0156%

	FC_3CIRG			FC_4CIRG		
($t,T,T+\delta$)	FC	MC	SE	FC	MC	SE
(0,1,1.5)	101.5601%	101.5855%	0.0368%	204.1894%	204.2528%	0.1438%
(0,2,2.5)	97.7516%	97.7789%	0.0352%	198.4813%	198.2691%	0.1569%
(0,3,3.5)	93.8237%	93.7891%	0.0336%	191.7518%	191.6992%	0.1612%
(0,4,4.5)	89.9498%	89.9635%	0.0325%	184.9873%	185.1965%	0.1652%
(0,5,5.5)	86.3292%	86.3034%	0.0309%	178.4444%	178.6016%	0.1679%
(0,10,10.5)	71.1653%	71.1760%	0.0258%	150.7148%	150.8163%	0.1673%
(0,15,15.5)	60.2162%	60.2380%	0.0218%	130.1908%	130.1983%	0.1683%
(0,20,20.5)	52.1325%	52.1312%	0.0190%	118.5521%	118.1314%	0.1770%
(0,25,25.5)	46.1628%	46.1707%	0.0166%	110.8434%	110.8827%	0.1922%

DOI: 10.0612/article-2

(0,30,30.5)	41.5718%	41.5984%	0.0152%	104.4679%	104.4739%	0.2038%

V. Conclusions

Four different types of CIRGOs and FCCIRGs have been developed via the cross-currency LMM. The guaranteed contracts with the underlying asset and the guaranteed interest rate denominated in different currencies have been analyzed, and the guaranteed rate is set relative to the level of the LIBOR rate. The pricing formulae derived are more consistent with market practice than those given in the previous researches. They can also be applied to both maturity-guarantees and multi-period guarantees with an arbitrary guarantee period δ. The derived pricing formulae represent the general formulae of the Margrabe (1978) type or the Black type in the framework of the cross-currency LMM and are easy for practical implementation. In addition, the pricing formulae have been shown numerically to be very accurate as compared with Monte-Carlo simulation. Pricing the guaranteed contracts with the derived formulae can be executed more efficiently than by adopting simulation, especially for the guaranteed contracts with a long duration such as life insurance or pension plans. Thus, the pricing formulae of FCCIRGs derived under the cross-currency LIBOR market model are more tractable and feasible for practical implementation.

References

Amin, K.I., and Jarrow, R., 1991, Pricing Foreign Currency Options under Stochastic Interest Rates. *Journal of International Money and Finance*, 10, 310–329.

Black, F., 1976, The Pricing of Commodity Contracts. *Journal of Financial Economics*, 3, 167–179.

Brace, A., Dun, T.A., and Barton, G.., 1998, Towards a Central Interest Rate Model. Paper presented at the Conference Global Derivatives'98.

Brace, A., Gatarek, D., and Musiela, M., 1997, The Market Model of Interest Rate Dynamics. *Mathematical Finance*, 7, 127-147.

Brace, A., and Womersley, R.S., 2000, Exact Fit to the Swaption Volatility Matrix Using Semidefinite Programming. Paper presented at the ICBI Global Derivatives Conference.

Boyle, P. P. and Hardy, M. R., 1997, Reserving for Maturity Guarantees: Two Approaches. *Insurance: Mathematics and Economics*, 21, 113-127.

Boyle, P. P., and Schwartz, E. S., 1977, Equilibrium Prices of Guarantees under Equity-Linked Contracts. *Journal of Risk and Insurance*, 44, 639-660.

Brennan, J. M. and Schwartz, E. S., 1976, The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee. *Journal of Financial Economics*, 3,195-213.

Brigo, D., and Mercurio, F., 2001, *Interest Rate Models: Theory and Practice*. Springer Verlag, Heidelberg.

Cox, J.C., Ingersoll J.E., and Ross, S.A., 1985, A Theory of the Term Structure of Interest Rates. *Econometrica*, 53, 385-407.

Ekern, S. and Persson, S. A., 1996, Exotic Unit-Linked Life Insurance Contracts. *The GENEVA Papers on Risk and Insurance Theory,* 21, 35–63.

Gatarek, D., 2003, Calibration of the LIBOR Market Model: Three Prescriptions. *Working paper*.

Grosen, A. and Jorgensen, P. L., 1997, Valuation of Early Exercisable Interest Rate Guarantees. *Journal of Risk and Insurance*, 64, 481-503.

Grosen, A. and Jorgensen, P. L., 2000, Fair Valuation of Life Insurance Liabilities: The

Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies. *Insurance: Mathematics and Economics*, 26, 37-57.

Hansen, M. and Miltersen, K. R., 2002, Minimum Rate of Return Guarantees: The Danish Case. *Scandinavian Actuarial Journal*, 4, 280-318.

Heath, D., Jarrow, R. and Morton A., 1992, Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica*, 60, 77-105.

Harrison, J.M., and Kreps, D.M., 1979, Martingales and Arbitrage in Multiperiod Securities Markets. *Journal of Economic Theory*, 20, 381-408.

Harrison, J.M., & Pliska, S.R., 1981, Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes and Their Applications*, 11, 215-260.

Hull, J., 2003, *Options, Futures and Other Derivatives (5th ed.)*, Prentice Hall, New Jersey.

Lindset, S., 2003, Pricing of Multi-period Rate of Return Guarantees. *Insurance: Mathematics and Economics*, 33, 629-644.

Lindset, S., 2004, Relative Guarantees. *The Geneva Papers on Risk and Insurance Theory*, 29, 187-209.

Margrabe, W., 1978, The Value of an Option to Exchange One Asset for Another. *The Journal of Finance*, 33, 177-186.

Miltersen, K. R. and Persson, S. A., 1999, Pricing Rate of Return Guarantee in a Heath-Jarrow-Morton Framework. *Insurance: Mathematics and Economics*, 25, 307–325.

Miltersen, K. R., Sandmann, K. and Sondermann, D., 1997, Closed Form Solutions for Term Structure Derivatives with Log-normal Interest Rates. *The Journal of Finance*, 52, 409-430.

Musiela, M., and Rutkowski, M., 1997, Continuous-time Term Structure Model: Forward Measure Approach. *Finance and Stochastics*, 4, 261-292.

Musiela, M., and M. Rutkowski, *2005*, *Martingale methods in financial modelling*, 2nd edn. (Springer Verlag, Heidelberg, Germany).

Pennacchi, G. G., 1999, The Value of Guarantees on Pension Fund Returns. *Journal of Risk and Insurance*, 66, 219-237.

Persson, S. A. and Aase, K. K., 1997, Valuation of the Minimum Guaranteed Return Embedded in Life Insurance Products. *Journal of Risk and Insurance*, 64, 599-617.

Rebonato, R., 1999, On the Simultaneous Calibration of Multifactor Lognormal Interest Rate Models to Black Volatilities and to the Correlation Matrix. *The Journal of Computational* Finance, 2, 5-27.

Rogers, C., 1996, Gaussian Errors. *Risk*, 9, 42-45.

Schlogl, E., 2002. A Multicurrency Extension of the Lognormal Interest Rate Market Models. *Finance and Stochastics*, 6, 173-188.

Schrager, D. F. and Pelsser, A. A., 2004, Pricing Rate of Return Guarantees in Regular Premium Unit Linked Insurance. *Insurance: Mathematics and Economics*, 35, 369-398.

Vasicek, O., 1977, An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, 5, 177-188.

Walker, K. L., 1992, *Guaranteed Investment Contracts: Risk Analysis and Portfolio Strategies*, Business One Irwin, Illinois.

Wu T. P. , Chen S. N., 2007, Cross-currency Equity Swaps with the BGM Model, *Journal of Derivatives*, Winter, 60-76.

Yang, S. S., Yueh, M. L. and Tang, C. H., 2008, Valuation of the Interest Rate Guarantee Embedded in Defined Contribution Pension Plans, *Insurance: Mathematics and Economics,* 42, 920-934.

DOI: 10.0612/article-2

Authors

Tsung-Yu Hsieh

Department of Banking and Finance, Tamkang University, New Taipei City 25137, Taiwan, 139709@mail.tku.edu.tw, tyhsieh@mail.tku.edu.tw

Chi-Hsun Chou

Department of Management, Fo Guang University, Yilan County 26247, Taiwan

Son-Nan Chen

Shanghai Advanced Institute of Finance, Shanghai Jiao Tong University, Shanghai P.R.China, 200030

*Corresponding author

Appendix A: Proof of Theorem 3.1

A.1 Proof of Equation (3.1.5)

By applying the martingale pricing method, the price of an C_1IRGO at time *t*, $0 \le t \le T \le T+\delta$, is derived as follows:

$$C_1IRGO(t) = N_d E^Q\left\{ e^{\left(-\int_t^{T+\delta} r_s d_s\right)}\left[\left(1+\delta L_d^\delta(T,T)\right) - \frac{S_f(T+\delta)}{S_f(T)}\right]^+ \middle| F_t \right\} \quad \text{(A.1)}$$

$$= N_d E^Q\left\{ \frac{P_d(T+\delta,T+\delta)/P_d(t,T+\delta)}{\beta_d(T+\delta)/\beta_d(t)} P_d(t,T+\delta)\left[\left(1+\delta L_d^\delta(T,T)\right) - \frac{S_f(T+\delta)}{S_f(T)}\right]^+ \middle| F_t \right\} \quad \text{(A.2)}$$

$$= N_d P_d(t,T+\delta) E^{T+\delta}\left\{ \left[\frac{P_d(T,T)}{P_d(T,T+\delta)} - \frac{S_f(T+\delta)}{S_f(T)}\right] \mathrm{I}_A \middle| F_t \right\} \quad \text{(A.3)}$$

where $1+\delta L_d^\delta(T,T) = \dfrac{P_d(T,T)}{P_d(T,T+\delta)}$, $A = \left\{ \dfrac{P_d(T,T)}{P_d(T,T+\delta)} > \dfrac{S_f(T+\delta)}{S_f(T)} \right\}$

$$= N_d P_d(t,T+\delta)\left\{ \underbrace{E^{T+\delta}\left[\frac{P_d(T,T)}{P_d(T,T+\delta)} \mathrm{I}_A \middle| F_t \right]}_{(A1)} - \underbrace{E^{T+\delta}\left[\frac{S_f(T+\delta)}{S_f(T)} \mathrm{I}_A \middle| F_t \right]}_{(A2)} \right\} \quad \text{(A.4)}$$

where

$E^Q(\cdot)$ denotes the expectation under the domestic martingale measure *Q*.

$E^{T+\delta}(\cdot)$ denotes the expectation under the domestic forward martingale measure $Q^{T+\delta}$ defined by the Radon-Nikodym derivative $dQ^{T+\delta}/dQ = \frac{P_d(T+\delta,T+\delta)/P_d(t,T+\delta)}{\beta_d(T+\delta)/\beta_d(t)}$.

I_A is an indicator function with $\begin{cases} 1, & if \quad \left(1+\delta L_d^\delta(T,T)\right) > S_f(T+\delta)/S_f(T) \\ 0, & otherwise \end{cases}$.

The dynamics of $S_f(T)$, $S_f(T+\delta)$, and $P_d(T,T)/P_d(T,T+\delta)$ are determined below.

$$S_f(T+\delta) = \frac{S_f(T+\delta)X(T+\delta)/P_d(T+\delta,T+\delta)}{X(T+\delta)P_f(T+\delta,T+\delta)/P_d(T+\delta,T+\delta)} = \frac{A(T+\delta)}{B(T+\delta)} \equiv Y(T+\delta)$$

$$S_f(T) = \frac{S_f(T)X(T)/P_d(T,T+\delta)}{X(T)P_f(T,T)/P_d(T,T+\delta)} = \frac{A(T)}{D(T)} \equiv Z(T)$$

$$\frac{P_d(T,T)}{P_d(T,T+\delta)} \equiv E(T)$$

We define each variable at time *t* as follows.

International Research Journal of Applied Finance ISSN 2229 – 6891
Vol. VI Issue – 12 December, 2015 www.irjaf.com
DOI: 10.0612/article-2

$$A(t)=S_f(t)X(t)/P_d(t,T+\delta) \tag{A-5}$$

$$B(t)=X(t)P_f(t,T+\delta)/P_d(t,T+\delta) \tag{A-6}$$

$$D(t)=X(t)P_f(t,T)/P_d(t,T+\delta) \tag{A-7}$$

$$E(t)=P_d(t,T)/P_d(t,T+\delta) \tag{A-8}$$

$$Y(t)=\frac{S_f(t)X(t)/P_d(t,T+\delta)}{X(t)P_f(t,T+\delta)/P_d(t,T+\delta)}=\frac{A(t)}{B(t)} \tag{A-9}$$

$$Z(t)=\frac{S_f(t)X(t)/P_d(t,T+\delta)}{X(t)P_f(t,T)/P_d(t,T+\delta)}=\frac{A(t)}{D(t)} \tag{A-10}$$

From proposition 2.2, the dynamics of (A-5) from (A-10) under the forward measure $Q^{T+\delta}$ can be obtained by Ito's Lemma as given below.

$$\frac{dA(t)}{A(t)}=\left[\underbrace{\sigma_{Sf}(t)+\sigma_X(t)+\overline{\sigma}_{P_d}(t,T+\delta)}_{\gamma_A(t)}\right]\cdot dW_t^{T+\delta}=\gamma_A(t)\cdot dW_t^{T+\delta} \tag{A-11}$$

$$\frac{dB(t)}{B(t)}=\left[\underbrace{\sigma_X(t)-\overline{\sigma}_{P_f}(t,T+\delta)+\overline{\sigma}_{P_d}(t,T+\delta)}_{\gamma_B(t)}\right]\cdot dW_t^{T+\delta}=\gamma_B(t)\cdot dW_t^{T+\delta} \tag{A-12}$$

$$\frac{dD(t)}{D(t)}=\left[\underbrace{\sigma_X(t)-\overline{\sigma}_{P_f}(t,T)+\overline{\sigma}_{P_d}(t,T+\delta)}_{\gamma_D(t)}\right]\cdot dW_t^{T+\delta}=\gamma_D(t)\cdot dW_t^{T+\delta} \tag{A-13}$$

$$\frac{dE(t)}{E(t)}=\left[\underbrace{-\overline{\sigma}_{P_d}(t,T)+\overline{\sigma}_{P_d}(t,T+\delta)}_{\gamma_E(t)}\right]\cdot dW_t^{T+\delta}=\gamma_E(t)\cdot dW_t^{T+\delta} \tag{A-14}$$

$$\begin{aligned}\frac{dY(t)}{Y(t)}=\frac{d\left[A(t)/B(t)\right]}{A(t)/B(t)}&=\left[\underbrace{-\gamma_B(t)\cdot(\gamma_A(t)-\gamma_B(t))}_{\overline{\mu_Y}(t)}\right]dt+\left[\underbrace{\gamma_A(t)-\gamma_B(t)}_{\psi_Y(t)}\right]\cdot dW_t^{T+\delta}\\&=\overline{\mu_Y}(t)dt+\psi_Y(t)\cdot dW_t^{T+\delta}\end{aligned} \tag{A-15}$$

DOI: 10.0612/article-2

$$\frac{dZ(t)}{Z(t)}=\frac{d\left[A(t)/D(t)\right]}{A(t)/D(t)}=\left[\underbrace{-\gamma_D(t)\cdot\left(\gamma_A(t)-\gamma_D(t)\right)}_{\overline{\mu_Z}(t)}\right]dt+\left[\underbrace{\gamma_A(t)-\gamma_D(t)}_{\psi_Z(t)}\right]\cdot dW_t^{T+\delta}$$
$$=\overline{\mu_Z}(t)dt+\psi_Z(t)\cdot dW_t^{T+\delta} \quad \text{(A-16)}$$

Solving the stochastic differential equations from (A-11) to (A-16), we obtain:

$$S_f(T+\delta)=Y(T+\delta)=\frac{A(T+\delta)}{B(T+\delta)}=\frac{A(t)}{B(t)}e^{\int_t^{T+\delta}\left[\overline{\mu_Y}(u)-\frac{1}{2}\left\|\psi_Y(u)\right\|^2\right]du+\int_t^{T+\delta}\psi_Y(u)\cdot dW_u^{T+\delta}} \quad \text{(A-17)}$$

$$S_f(T)=Z(T)=\frac{A(T)}{D(T)}=\frac{A(t)}{D(t)}e^{\int_t^{T}\left[\overline{\mu_Z}(u)-\frac{1}{2}\left\|\psi_Z(u)\right\|^2\right]du+\int_t^{T}\psi_Z(u)\cdot dW_u^{T+\delta}} \quad \text{(A-18)}$$

$$\frac{S_f(T+\delta)}{S_f(T)}=\frac{Y(T+\delta)}{Z(T)}=\frac{A(T+\delta)/B(T+\delta)}{A(T)/D(T)}$$
$$=\frac{P_f(t,T)}{P_f(t,T+\delta)}e^{\int_t^T\theta_{11}(u)du+\int_T^{T+\delta}\theta_{12}(u)du}e^{-\frac{1}{2}\int_t^T\left(\left\|\psi_Y(u)-\psi_Z(u)\right\|^2\right)du+\int_t^T\left[\psi_Y(u)-\psi_Z(u)\right]\cdot dW_u^{T+\delta}-\frac{1}{2}\int_T^{T+\delta}\left\|\psi_Y(u)\right\|^2du+\int_T^{T+\delta}\psi_Y(u)\cdot dW_u^{T+\delta}} \quad \text{(A-19)}$$

where

$$\theta_{11}(t)=\left[\overline{\mu_Y}(t)-\overline{\mu_Z}(t)\right]-\left[\psi_Y(t)\cdot\psi_Z(t)-\left\|\psi_Z(t)\right\|^2\right]$$
$$=\left[\sigma_X(t)-\overline{\sigma}_{P_f}(t,T+\delta)+\overline{\sigma}_{P_d}(t,T+\delta)\right]\cdot\left[\overline{\sigma}_{P_f}(t,T)-\overline{\sigma}_{P_f}(t,T+\delta)\right] \quad \text{(A-20)}$$

$$\theta_{12}(t)=\overline{\mu_Y}(t)$$
$$=\left[\sigma_X(t)-\overline{\sigma}_{P_f}(t,T+\delta)+\overline{\sigma}_{P_d}(t,T+\delta)\right]\cdot\left[-\overline{\sigma}_{P_f}(t,T+\delta)-\sigma_{S_f}(t)\right] \quad \text{(A-21)}$$

$$\frac{P_d(T,T)}{P_d(T,T+\delta)}=E(T)=\frac{P_d(t,T)}{P_d(t,T+\delta)}e^{\int_t^T-\frac{1}{2}\left\|\gamma_E(u)\right\|^2dt+\int_t^T\gamma_E(u)\cdot dW_u^{T+\delta}} \quad \text{(A-22)}$$

Part (A1) and (A2) are solved, respectively, as follows.

$$(A1)=\frac{P_d(t,T)}{P_d(t,T+\delta)}E_t^{T+\delta}\left[e^{-\frac{1}{2}\int_t^T\left\|\gamma_E(u)\right\|^2du+\int_t^T\gamma_E(u)\cdot dW_u^{T+\delta}}I_A\right]$$

$$=\frac{P_d(t,T)}{P_d(t,T+\delta)}E_t^{T+\delta}\left[\frac{dR_1}{dQ^{T+\delta}}I_A\right] \qquad where\ \frac{dR_1}{dQ^{T+\delta}}=e^{-\frac{1}{2}\int_t^T\left\|\gamma_E(u)\right\|^2du+\int_t^T\gamma_E(u)\cdot dW_u^{T+\delta}}$$

$$=\frac{P_d(t,T)}{P_d(t,T+\delta)}P_r^{R_1}\left(\frac{P_d(T,T)}{P_d(T,T+\delta)}>\frac{S_f(T+\delta)}{S_f(T)}\right)=\frac{P_d(t,T)}{P_d(t,T+\delta)}N(-d_{12})$$
$$=\left[1+\delta L_d^{\delta}(t,T)\right]N(-d_{12}) \quad \text{(A-23)}$$

where the measure R_1 is defined by the Radon-Nikodym derivative $dR_1 / dQ^{T+\delta}$.

We next show that $E_t^{T+\delta}\left[\frac{dR_1}{dQ^{T+\delta}} I_A\right] = P_r^{R_1}\left(\frac{P_d(T,T)}{P_d(T,T+\delta)} > \frac{S_f(T+\delta)}{S_f(T)}\right) = N(-d_{12})$

From the Radon-Nikodym derivative, we know that the relation of the Brownian motions under the different measures can be shown as:

For time interval $[t,T]: dW_t^{T+\delta} = dW_t^{R_1} + \gamma_E(t)dt$ (A-24)

For time interval $[T,T+\delta]: dW_t^{T+\delta} = dW_t^{R_1}$ (A-25)

Substituting (A-24) and (A-25) into (A-19) and (A-22), we can obtain the dynamics under measure R_1.

$$\frac{S_f(T+\delta)}{S_f(T)} = \frac{P_f(t,T)}{P_f(t,T+\delta)} e^{\int_t^T \theta_{11}(u)du + \int_T^{T+\delta}\theta_{12}(u)du} e^{-\frac{1}{2}\int_t^T\left(\|\psi_Y(u)-\psi_Z(u)\|^2 - 2[\psi_Y(u)-\psi_Z(u)]\cdot\gamma_E(u)\right)du + \int_t^T[\psi_Y(u)-\psi_Z(u)]\cdot dW_u^{R_1} - \frac{1}{2}\int_T^{T+\delta}\|\psi_Y(u)\|^2 du + \int_T^{T+\delta}\psi_Y(u)\cdot dW_u^{R_1}} \quad \text{(A-26)}$$

$$\frac{P_d(T,T)}{P_d(T,T+\delta)} = \frac{P_d(t,T)}{P_d(t,T+\delta)} e^{\frac{1}{2}\int_t^T \|\gamma_E(u)\|^2 du + \int_t^T \gamma_E(u)\cdot dW_u^{R_1}} \quad \text{(A-27)}$$

By inserting (A-26) and (A-27) into $P_r^{R_1}[\cdot]$, the probability can be obtained after rearrangement as follows:

$$P_r^{R_1}\left(\frac{P_d(T,T)}{P_d(T,T+\delta)} > \frac{S_f(T+\delta)}{S_f(T)}\right) = N(-d_{12})$$

$$d_{12} = \frac{\ln\left[\frac{1+\delta L_f^{\delta}(t,T)}{1+\delta L_d^{\delta}(t,T)}\right] + \int_t^T \theta_{11}(u)du + \int_T^{T+\delta}\theta_{12}(u)du - \frac{1}{2}V_1^2}{V_1} \quad \text{(A-28)}$$

$$V_1^2 = \int_t^T \left(\|\theta_{13}(u)\|^2\right)du + \int_T^{T+\delta}\|\theta_{14}(u)\|^2 du \quad \text{(A-29)}$$

$$\theta_{13}(t) = \psi_Y(t) - \psi_Z(t) - \gamma_E(t) = \left[\bar{\sigma}_{P_f}(t,T+\delta) - \bar{\sigma}_{P_f}(t,T) + \bar{\sigma}_{P_d}(t,T) - \bar{\sigma}_{P_d}(t,T+\delta)\right] \quad \text{(A-30)}$$

$$\theta_{14}(t) = \psi_Y(t) = \sigma_{S_f}(t) + \bar{\sigma}_{P_f}(t,T+\delta) \quad \text{(A-31)}$$

The procedure to solve (A2) is similar to that of (A1).

DOI: 10.0612/article-2

$$(A2)=\frac{P_f(t,T)}{P_f(t,T+\delta)}e^{\int_t^T \theta_{11}(u)du+\int_T^{T+\delta}\theta_{12}(u)du}E_t^{T+\delta}\left[\frac{dR_2}{dQ^{T+\delta}}I_A\right]$$

$$\text{where}\quad \frac{dR_2}{dQ^{T+\delta}}=e^{\int_t^T -\frac{1}{2}\left(\|\psi_Y(u)-\psi_Z(u)\|^2\right)du+\int_t^T[\psi_Y(u)-\psi_Z(u)]\cdot dW_u^{T+\delta}+\int_T^{T+\delta}-\frac{1}{2}\|\psi_Y(u)\|^2du+\int_T^{T+\delta}\psi_Y(u)\cdot dW_u^{T+\delta}} \quad \text{(A-32)}$$

$$=\frac{P_f(t,T)}{P_f(t,T+\delta)}e^{\int_t^T \theta_{11}(u)du+\int_T^{T+\delta}\theta_{12}(u)du}N(-d_{11})$$

$$=\left[1+\delta L_f^{\delta}(t,T)\right]e^{\int_t^T \theta_{11}(u)du+\int_T^{T+\delta}\theta_{12}(u)du}N(-d_{11})$$

We next show that

$$E_t^{T+\delta}\left[\tfrac{dR_2}{dQ^{T+\delta}}I_A\right]=P_r^{R_2}\left(\frac{P_d(T,T)}{P_d(T,T+\delta)}>\frac{S_f(T+\delta)}{S_f(T)}\right)=N(-d_{11})$$

From the Radon-Nikodym derivative, we can obtain the relations as below :

For time interval $[t,T]: dW_t^{T+\delta}=dW_t^{R_2}+\left[\psi_Y(t)-\psi_Z(t)\right]dt$ (A-33)

For time interval $[T,T+\delta]: dW_t^{T+\delta}=dW_t^{R_2}+\psi_Y(t)dt$ (A-34)

Substituting (A-33) and (A-34) into (A-22) and (A-25), we obtain the dynamics under the measure R_2.

$$\frac{S_f(T+\delta)}{S_f(T)}$$

$$=\frac{P_f(t,T)}{P_f(t,T+\delta)}e^{\int_t^T \theta_{11}(u)du+\int_T^{T+\delta}\theta_{12}(u)du}e^{\frac{1}{2}\int_t^T\left(\|\psi_Y(u)-\psi_Z(u)\|^2\right)du+\int_t^T[\psi_Y(u)-\psi_Z(u)]\cdot dW_u^{R_2}+\frac{1}{2}\int_T^{T+\delta}\|\psi_Y(u)\|^2du+\int_T^{T+\delta}\psi_Y(u)\cdot dW_u^{R_2}} \quad \text{(A-35)}$$

$$\frac{P_d(T,T)}{P_d(T,T+\delta)}=\frac{P_d(t,T)}{P_d(t,T+\delta)}e^{-\frac{1}{2}\int_t^T\left(\|\gamma_E(u)\|^2-2\gamma_E(u)\cdot[\psi_Y(u)-\psi_Z(u)]\right)du+\int_t^T\gamma_E(u)\cdot dW_u^{R_2}} \quad \text{(A-36)}$$

Inserting (A-35) and (A-36) into $P_r^{R_2}[\cdot]$ and rearranging them, we obtain

$$P_r^{R_1}\left(\frac{P_d(T,T)}{P_d(T,T+\delta)}>\frac{S_f(T+\delta)}{S_f(T)}\right)=N(-d_{11}),\qquad \text{where}\quad d_{11}=d_{12}+V_1$$

By combining (A-23) with (A-32), equation (3.1.5) of Theorem 3.1.1 is obtained.

A.2 Proof of Equation (3.1.6)

By using the martingale pricing method, FC_1IRGs can be valued as follows.

$$FC_1(t)=E^Q\left[e^{-\int_t^{T+\delta}r_s d_s}FC_1(T+\delta)\right]$$

$$= N_d \left\{ \underbrace{E^Q\left[e^{-\int_t^{T+\delta} r_s d_s} \frac{S_f(T+\delta)}{S_f(T)} \right]}_{(A3)} + \underbrace{E^Q\left[e^{-\int_t^{T+\delta} r_s d_s} \left(\left(1+\delta L_d^\delta(T,T)\right) - \frac{S_f(T+\delta)}{S_f(T)} \right)^+ \right]}_{(A4)} \right\}$$

By equation (A-19) and the stochastic calculus, we obtain the result as below.

$$\text{(A3)} = P_d\left(t,T+\delta\right) E^{T+\delta}\left[\frac{S_f(T+\delta)}{S_f(T)} \right] = P_d\left(t,T+\delta\right)\left(1+\delta L_f^\delta(T,T)\right) e^{\int_t^T \theta_{11}(u)du + \int_T^{T+\delta} \theta_{12}(u)du} \quad \text{(A-37)}$$

(A4) is equal to the pricing formula of C_1IRGOs in Theorem 3.1.1, i.e., equation (3.1.5). Hence, combining (A-37) and (3.1.5), (3.1.6) in Theorem 3.1.2 can be obtained.

Appendix B: Proof of Theorem 3.2.1

B.1 Proof of Equation (3.2.4)

By applying the martingale pricing method, the price of an C$_2$IRGO at time t, $0 \le t \le T \le T+\delta$, is derived as follows:

$$C_2IRGO(t) = N_d E^Q \left\{ e^{\left(-\int_t^{T+\delta} r_s d_s\right)} \left[\left(1+\delta L_f^\delta(T,T)\right) - \frac{S_d(T+\delta)}{S_d(T)} \right]^+ \middle| F_t \right\} \quad \text{(B-1)}$$

$$= N_d P_d(t,T+\delta) E^{T+\delta} \left\{ \left[\left(1+\delta L_f^\delta(T,T)\right) - \frac{S_d(T+\delta)}{S_d(T)} \right] \mathrm{I}_A \middle| F_t \right\} \quad \text{(B-2)}$$

$$= N_d P_d(t,T+\delta) \left\{ \underbrace{E^{T+\delta}\left[\frac{P_f(T,T)}{P_f(T,T+\delta)} \mathrm{I}_A \middle| F_t \right]}_{(B1)} - \underbrace{E^{T+\delta}\left[\frac{S_d(T+\delta)}{S_d(T)} \mathrm{I}_A \middle| F_t \right]}_{(B2)} \right\} \quad \text{(B-3)}$$

$$where \quad A = \left\{ \frac{P_f(T,T)}{P_f(T,T+\delta)} > \frac{S_d(T+\delta)}{S_d(T)} \right\}, \qquad \left(1+\delta L_f^\delta(T,T)\right) = \frac{P_f(T,T)}{P_f(T,T+\delta)}$$

The dynamics of $S_d(T)$, $S_d(T+\delta)$ and $P_f(T,T)/P_f(T,T+\delta)$ are determined below.

$$S_d(T+\delta) = \frac{S_d(T+\delta)}{P_d(T+\delta,T+\delta)} \equiv M(T+\delta) \quad \text{(B-4)}$$

$$S_d(T) = \frac{S_d(T)/P_d(T,T+\delta)}{P_d(T,T)/P_d(T,T+\delta)} = \frac{M(T)}{E(T)} \equiv N(T) \quad \text{(B-5)}$$

$$\frac{S_d(T+\delta)}{S_d(T)} = \frac{M(T+\delta)}{M(T)/E(T)} = E(T)\frac{M(T+\delta)}{M(T)} \quad \text{(B-6)}$$

$$\frac{P_f(T,T)}{P_f(T,T+\delta)} = \frac{X(T)P_f(T,T)/P_d(T,T+\delta)}{X(T)P_f(T,T+\delta)/P_d(T,T+\delta)} = \frac{D(T)}{B(T)} \equiv V(T) \quad \text{(B-7)}$$

Hence, each variable at time t is defined as follows.

$$M(t) = S_d(t)/P_d(t,T+\delta) \quad \text{(B-8)}$$

$$E(t) = P_d(t,T)/P_d(t,T+\delta) \quad \text{(B-9)}$$

$$N(t) = \frac{S_d(t)/P_d(t,T+\delta)}{P_d(t,T)/P_d(t,T+\delta)} = \frac{M(t)}{E(t)} \quad \text{(B-10)}$$

$V(t)=D(t)/B(t)$, $B(t)$ and $D(t)$ are defined as (A-6) and (A-7) in the appendix A. (B-11)

From proposition 2.2, the dynamics of (B-8) from (B-11) under the forward measure $Q^{T+\delta}$ can be obtained by using Ito's Lemma and given below.

$$\frac{dM(t)}{M(t)}=\left[\underbrace{\sigma_{Sd}(t)+\bar{\sigma}_{P_d}(t,T+\delta)}_{\gamma_M(t)}\right]\cdot dW_t^{T+\delta}=\gamma_M(t)\cdot dW_t^{T+\delta} \tag{B-12}$$

$$\frac{dE(t)}{E(t)}=\left[\underbrace{-\bar{\sigma}_{P_d}(t,T)+\bar{\sigma}_{P_d}(t,T+\delta)}_{\gamma_E(t)}\right]\cdot dW_t^{T+\delta}=\gamma_E(t)\cdot dW_t^{T+\delta} \tag{B-13}$$

$$\frac{dB(t)}{B(t)}=\left[\underbrace{\sigma_X(t)-\bar{\sigma}_{P_f}(t,T+\delta)+\bar{\sigma}_{P_d}(t,T+\delta)}_{\gamma_B(t)}\right]\cdot dW_t^{T+\delta}=\gamma_B(t)\cdot dW_t^{T+\delta} \text{ as defined in } \tag{A-12}$$

$$\frac{dD(t)}{D(t)}=\left[\underbrace{\sigma_X(t)-\bar{\sigma}_{P_f}(t,T)+\bar{\sigma}_{P_d}(t,T+\delta)}_{\gamma_D(t)}\right]\cdot dW_t^{T+\delta}=\gamma_D(t)\cdot dW_t^{T+\delta} \text{ as defined in } \tag{A-13}$$

$$\begin{aligned}\frac{dV(t)}{V(t)}&=\left[\underbrace{-\gamma_B(t)\cdot\gamma_D(t)+\left\|\gamma_B(t)\right\|^2}_{\overline{\mu_V}(t)}\right]dt+\left[\underbrace{\gamma_D(t)-\gamma_B(t)}_{\psi_V(t)}\right]\cdot dW_t^{T+\delta}\\&=\overline{\mu_V}(t)\,dt+\psi_V(t)\cdot dW_t^{T+\delta}\end{aligned} \tag{B-14}$$

Solving the stochastic differential equations (A-12), (A-13) and from (B-12) to (B-14), we obtain:

$$M(T+\delta)=M(t)e^{-\frac{1}{2}\int_t^{T+\delta}\left\|\gamma_M(u)\right\|^2du+\int_t^{T+\delta}\gamma_M(u)\cdot dW_u^{T+\delta}} \tag{B-15}$$

$$M(T)=M(t)e^{-\frac{1}{2}\int_t^{T}\left\|\gamma_M(u)\right\|^2du+\int_t^{T}\gamma_M(u)\cdot dW_u^{T+\delta}} \tag{B-16}$$

$$E(T)=E(t)e^{-\frac{1}{2}\int_t^{T}\left\|\gamma_E(u)\right\|^2du+\int_t^{T}\gamma_E(u)\cdot dW_u^{T+\delta}} \tag{B-17}$$

$$\begin{aligned}\frac{S_d(T+\delta)}{S_d(T)}&=E(T)\frac{M(T+\delta)}{M(T)}\\&=\frac{P_d(t,T)}{P_d(t,T+\delta)}e^{-\frac{1}{2}\int_t^{T}\left\|\gamma_E(u)\right\|^2du+\int_t^{T}\gamma_E(u)\cdot dW_u^{T+\delta}}e^{-\frac{1}{2}\int_T^{T+\delta}\left\|\gamma_M(u)\right\|^2du+\int_T^{T+\delta}\gamma_M(u)\cdot dW_u^{T+\delta}}\end{aligned} \tag{B-18}$$

$$\frac{P_f(T,T)}{P_f(T,T+\delta)}=V(T)=\frac{P_f(t,T)}{P_f(t,T+\delta)}e^{\int_t^T\left[\overline{\mu_V}(u)-\frac{1}{2}\left\|\psi_V(u)\right\|^2\right]du+\int_t^T\psi_V(u)\cdot dW_u^{T+\delta}} \tag{B-19}$$

Part (B-I) and (B-II) are solved, respectively, as follows.

$$\begin{aligned}(B1) &= \frac{P_f(t,T)}{P_f(t,T+\delta)} e^{\int_t^T \overline{\mu_V}(u)du} E_t^{T+\delta}\left[e^{-\frac{1}{2}\int_t^T \|\psi_V(u)\|^2 du + \int_t^T \psi_V(u)\cdot dW_u^{T+\delta}} I_A \right] \\
&= \frac{P_f(t,T)}{P_f(t,T+\delta)} e^{\int_t^T \overline{\mu_V}(u)du} E_t^{T+\delta}\left[\frac{dR_1}{dQ^{T+\delta}} I_A \right] \quad where \quad \frac{dR_1}{dQ^{T+\delta}} = e^{-\frac{1}{2}\int_t^T \|\psi_V(u)\|^2 du + \int_t^T \psi_V(u)\cdot dW_u^{T+\delta}} \\
&= \frac{P_f(t,T)}{P_f(t,T+\delta)} e^{\int_t^T \overline{\mu_V}(u)du} P_r^{R_2}\left(\frac{P_f(T,T)}{P_f(T,T+\delta)} > \frac{S_d(T+\delta)}{S_d(T)} \right) \\
&= \frac{P_f(t,T)}{P_f(t,T+\delta)} e^{-\int_t^T \theta_{21}(u)du} N(-d_{22}) \quad where \quad \theta_{21}(t) = -\overline{\mu_V}(t) \\
&= \left[1+\delta L_f^\delta(t,T)\right] e^{-\int_t^T \theta_{21}(u)du} N(-d_{22})\end{aligned} \tag{B-20}$$

where the measure R_1 is defined by the Radon-Nikodym derivative $dR_1 / dQ^{T+\delta}$.

We next show that $E_t^{T+\delta}\left[\frac{dR_1}{dQ^{T+\delta}} I_A \right] = P_r^{R_1}\left(\frac{P_f(T,T)}{P_f(T,T+\delta)} > \frac{S_d(T+\delta)}{S_d(T)} \right) = N(-d_{22})$

From the Radon-Nikodym derivative, we know that the relation of the Brownian motions under different measures can be shown as:

For time interval $[t,T]: dW_t^{T+\delta} = dW_t^{R_1} + \psi_V(t)dt$ (B-21)

For time interval $[T,T+\delta]: dW_t^{T+\delta} = dW_t^{R_1}$ (B-22)

Substituting (B-21) and (B-22) into (B-18) and (B-19), we can obtain the dynamics under measure R_1.

$$\frac{S_d(T+\delta)}{S_d(T)} = \frac{P_d(t,T)}{P_d(t,T+\delta)} e^{-\frac{1}{2}\int_t^T \left[\|\gamma_E(u)\|^2 - 2\gamma_E(u)\cdot\psi_V(u)\right]du + \int_t^T \gamma_E(u)\cdot dW_u^{R_1} - \frac{1}{2}\int_T^{T+\delta} \|\gamma_M(u)\|^2 du + \int_T^{T+\delta} \gamma_M(u)\cdot dW_u^{R_1}} \tag{B-23}$$

$$\frac{P_f(T,T)}{P_f(T,T+\delta)} = \frac{P_f(t,T)}{P_f(t,T+\delta)} e^{\int_t^T \left[\overline{\mu_V}(u) + \frac{1}{2}\|\psi_V(u)\|^2\right]du + \int_t^T \psi_V(u)\cdot dW_u^{R_1}} \tag{B-24}$$

By inserting (B-23) and (B-24) into $P_r^{R_1}[\cdot]$, the probability can be obtained after rearrangement as follows:

$$P_r^{R_1}\left(\frac{P_f(T,T)}{P_f(T,T+\delta)} > \frac{S_d(T+\delta)}{S_d(T)} \right) = N(-d_{22})$$

$$d_{22} = \frac{\ln\left[\frac{1+\delta L_d^\delta(t,T)}{1+\delta L_f^\delta(t,T)}\right] + \int_t^T \theta_{21}(u)du - \frac{1}{2}V_2^2}{V_2} \tag{B-25}$$

$$V_2^2 = \int_t^T \left(\left\|\theta_{22}(u)\right\|^2\right) du + \int_T^{T+\delta} \left\|\theta_{23}(u)\right\|^2 du \tag{B-26}$$

$$\theta_{21}(t) = -\overline{\mu_V}(t) = \left[\sigma_X(t) - \overline{\sigma}_{P_f}(t, T+\delta) + \overline{\sigma}_{P_d}(t, T+\delta)\right] \cdot \left[\overline{\sigma}_{P_f}(t, T+\delta) - \overline{\sigma}_{P_f}(t, T)\right] \tag{B-27}$$

$$\theta_{22}(t) = \gamma_E(t) - \psi_V(t) = \left[\overline{\sigma}_{P_d}(t, T+\delta) - \overline{\sigma}_{P_d}(t, T) + \overline{\sigma}_{P_f}(t, T) - \overline{\sigma}_{P_f}(t, T+\delta)\right] \tag{B-28}$$

$$\theta_{23}(t) = \gamma_M(t) = \left[\sigma_{S_d}(t) + \overline{\sigma}_{P_d}(t, T+\delta)\right] \tag{B-29}$$

Since the procedure to solve (B2) is similar to that of (B1), we present the result without showing the derivation processes.[8]

$$(B2) = \frac{P_d(t,T)}{P_d(t,T+\delta)} E_t^{T+\delta}\left[\frac{dR_2}{dQ^{T+\delta}} I_A\right]$$

$$\text{where} \qquad \frac{dR_2}{dQ^{T+\delta}} = e^{-\frac{1}{2}\int_t^T \|\gamma_E(u)\|^2 du + \int_t^T \gamma_E(u)\cdot dW_u^{T+\delta}} e^{-\frac{1}{2}\int_T^{T+\delta} \|\gamma_M(u)\|^2 du + \int_T^{T+\delta} \gamma_M(u)\cdot dW_u^{T+\delta}}$$

$$= \frac{P_d(t,T)}{P_d(t,T+\delta)} P_r^{R_2}\left(\frac{P_f(T,T)}{P_f(T,T+\delta)} > \frac{S_d(T+\delta)}{S_d(T)}\right) = \frac{P_d(t,T)}{P_d(t,T+\delta)} N(-d_{21}) \tag{B-30}$$

$$= \left[1 + \delta L_d^\delta(t,T)\right] N(-d_{21})$$

$$d_{21} = d_{22} + V_2$$

By combining (B-20) with (B-30), equation (3.1.5) of Theorem 3.2.4 is obtained.

B.2 Proof of Equation (3.2.5)

By using the martingale pricing method, FC_2IRGs can be valued as follows.

$$FC_2(t) = E^Q\left[e^{-\int_t^{T+\delta} r_s d_s} FC_2(T+\delta)\right]$$

$$= N_d \left\{ \underbrace{E^Q\left[e^{-\int_t^{T+\delta} r_s d_s} \frac{S_d(T+\delta)}{S_d(T)}\right]}_{(B3)} + \underbrace{E^Q\left[e^{-\int_t^{T+\delta} r_s d_s}\left(\left(1+\delta L_f^\delta(T,T)\right) - \frac{S_d(T+\delta)}{S_d(T)}\right)^+\right]}_{(B4)} \right\}$$

By equation (B-21) and the stochastic calculus, we obtain the result below.

$$(\text{B3}) = P_d(t,T+\delta) E^{T+\delta}\left[\frac{S_d(T+\delta)}{S_d(T)}\right] = P_d(t,T+\delta)\left(1+\delta L_d^\delta(T,T)\right) \tag{B-31}$$

(B4) is equal to the pricing formula of C_2IRGOs in Theorem 3.2.1, i.e., equation (3.2.4). Hence, combining (B-31) and (3.2.4), (3.2.5) in Theorem 3.2.2, the final result can be obtained.

Appendix C: Share of Unit-Linked Contracts in Total Life Premium

Table C1: Share of Unit-Linked Contracts in Total Life Premium

	Life Premium (millions of euros)		Share-UL (%)†	
Country	2009	2010	2009	2010
Belgium	18,404	19,141	9.04%	10.70%
Bulgaria	103	115	5.28%	5.62%
Switzerland	19,484	21,828	9.51%	10.13%
Czech Republic	2,044	2,601	39.11%	46.80%
Germany	81,371	87,165	13.91%	13.48%
Denmark	14,342	14,938	25.04%	35.18%
Estonia	133	182	45.25%	61.52%
Spain	29,074	27,297	14.77%	17.44%
Finland	2,847	4,570	56.76%	56.04%
France	137,923	143,837	13.02%	13.39%
United Kingdom	155,417	152,583	17.00%	14.94%
Croatia	339	337	6.59%	6.79%
Hungary	1,466	1,606	57.21%	61.35%
Italy	81,116	90,102	12.00%	17.10%
Malta	193	224	12.95%	15.18%
Netherlands	24,381	21,573	34.04%	43.00%
Norway	7,140	8,382	20.04%	19.99%
Poland	6,982	7,848	21.37%	25.89%
Portugal	9,876	12,103	29.22%	22.04%
Sweden	18,134	22,203	35.94%	34.38%
Slovenia	630	656	58.57%	60.37%
Romania	384	214	n.a. ‡	41.69%
Cyprus	353	375	n.a.	n.a.
Latvia	28	n.a.	13.50%	n.a.
Greece	2,202	n.a.	n.a.	n.a.
Others	19,802	20,795	n.a.	n.a.
Total	634,169	660,676	15.96%	16.98%

Source: European Insurance and Reinsurance Federation.

† "Share-UL" represents the share of unit-linked contracts in total life premiums.

‡ "n.a." denotes "not available."

Notes

[1] Examples of such contracts can be observed in pension plans and unit-linked life insurance contracts. Countries that provide pension plans with a stochastic guaranteed rate include Chile, Colombia, Peru, and Argentina (see, e.g., Pennacchi 1999; Lindset 2004). Ekern and Persson (1996) analyze a number of unit-linked contracts with stochastic guaranteed rates. Table C1 in Appendix C shows the statistics regarding the unit-linked products provided by the European Insurance and Reinsurance Federation (CEA). From the statistics, the European life insurance market in 2010 was characterized by a significant rise in the percentage share of unit-linked contracts among all life premiums.

[2] More details can be seen in Harrison and Kreps (1979), Amin and Jarrow (1991), Schlogl (2002), Musiela and Rutkowski (2005), and Wu and Chen (2007).

[3] The filtration $\{F_t\}_{t\in[0,\tau]}$ is right continuous and F_0 contains all the Q-null sets of F.

[4] See AJ (1991) for more details regarding the regularity conditions.

[5] The results for multi-period guarantees are available from the authors upon request.

[6] The result is available upon request from the authors.

[7] The result is available upon request from the authors.

[8] See Rebonato (1999) and Wu and Chen (2007) for more details.

[9] Rebonato (1999) shows that his procedure represents a general calibration method without a constraint on the number of factors. For the purpose of demonstration, we employ a four-factor framework (m = 4), which is one of the cases demonstrated by Rebonato (1999), to implement numerical analyses. More details can be seen in Rebonato (1999).

[10] All data are drawn and computed from the DataStream database. All the market data associated with the domestic and foreign stock indexes, the exchange rates, domestic and foreign cap volatilities in the U.S. and U.K. markets, and initial forward LIBOR rates are available upon request from the authors.