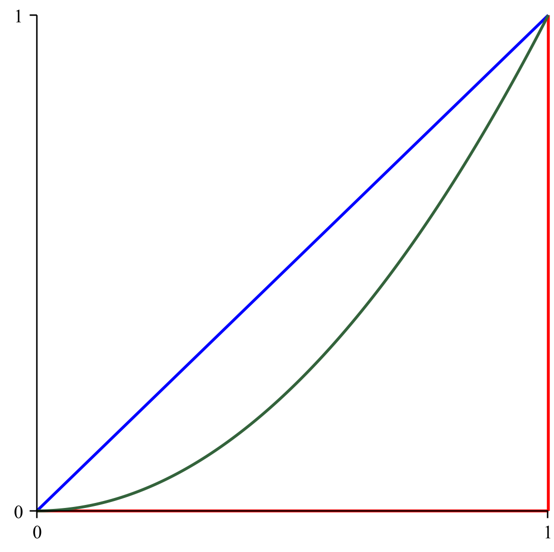
Calculus 3 - Line Integrals Over Conservative Vector Fields

Consider the line integral

$$\int_C 2xydx + x^2dy \tag{1}$$

where C is the following:

1. $y = x^2$ from $(0,0)$ to $(1,1)$.
2. $y = x$ from $(0,0)$ to $(1,1)$.
3. $y = 0$ from $(0,0)$ to $(1,0)$ then along $x = 1$ from $(1,0)$ to $(1,1)$.



Soln.

As there are three sets of curves, we do each problem separately.

C_1: Since $y = x^2$, then $dy = 2x\,dx$ and our line integral becomes

$$\int_0^1 2x \cdot x^2dx + 2x^2 \cdot 2xdx = \int_0^1 4x^3\,dx = x^4\Big|_0^1 = 1. \tag{2}$$

C_2: Since $y = x$, then $dy = dx$ and our line integral becomes

$$\int_0^1 2x \cdot x dx + x^2 \cdot dx = \int_0^1 3x^2\, dx = x^3\Big|_0^1 = 1. \tag{3}$$

C_3: Along $y = 0$, then $dy = 0$ so the line integral is zero. Along $x = 1$, $dx = 0$ and our line integral becomes

$$\int_0^1 dy = y\Big|_0^1 = 1. \tag{4}$$

Interesting! Along three different paths, we get the same answer. Suppose we change the line integral slightly, say

$$\int\limits_C xydx + x^2dy \tag{5}$$

we obtain

1. $\displaystyle\int\limits_{C_1} xydx + x^2dy = \int_0^1 3x^3\, dx = \frac{3}{4}.$

2. $\displaystyle\int\limits_{C_2} xydx + x^2dy = \int_0^1 2x^2\, dx = \frac{2}{3}.$

3. $\displaystyle\int\limits_{C_3} xydx + x^2dy = \int_0^1 dy = 1.$

three different answers. So what's special about the first line integral and, in particular, the vector field

$$\vec{F} = \langle 2xy, x^2 \rangle? \tag{6}$$

Answer − It's conservative. It is an easy matter to show that

$$f = x^2 y + c \tag{7}$$

and we recall from differentials that

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \quad \Rightarrow \quad d\left(x^2 y\right) = 2xy\, dx + x^2\, dy. \tag{8}$$

Now consider the (1) again so

$$\int_C 2xydx + x^2 dy = \int_C d\left(x^2 y\right) = x^2 y\,\Big|_{(0,0)}^{(1,1)} = 1 \tag{9}$$

Fundemental Theorem of Line Integrals

Let C be a smooth curve defined by the vector function $\vec{r}(t)$. Let $\vec{F}$ be a continuous vector field with scalar potential f then

$$\int_C \vec{F} \cdot d\vec{r} = f\,\Big|_A^B = f(B) - f(A) \tag{10}$$

where A and B are the initial and terminal points along the curve C.

Example 1. Evaluate the following using the fundamental thm of line integrals.

$$\int_C 2(x+y)dx + 2(x+y)dy \tag{11}$$

where C is a smooth curve from $(-1, 0)$ to $(3, 2)$.

Soln. First we show the vector field is conservative. Here

$$P = 2(x+y), \quad Q = 2(x,y) \text{ and } \quad Q_x = 2 = P_y \tag{12}$$

so yes, it is. Next we find f.

$$f_x = P = 2x + 2y, \quad f_y = Q = 2x + 2y \tag{13}$$

so

$$f = x^2 + 2xy + y^2 + c \tag{14}$$

(although we neglect the c) so

$$\int_C 2(x+y)dx + 2(x+y)dy = x^2 + 2xy + y^2 \Big|_{(-1,0)}^{(3,2)} = 24. \tag{15}$$

Example 2. Evaluate the following line integral.

$$\int_C yz\,dx + xz\,dy + (xy+1)\,dz \tag{16}$$

where C is a smooth curve

$$x = \cos t, \quad y = \sin t, \quad z = t, \quad 0 \le t \le 2\pi. \tag{17}$$

Soln. We will evaluate this first directly and then using the fundamental theorem.

Directly. We calculate differentials so

$$dx = -\sin t\,dt, \quad dy = \cos t\,dt, \quad dz = dt \tag{18}$$

Eqn. (16) becomes

$$\begin{aligned}&\int_0^{2\pi} -t\sin^2 t dt + t\cos^2 t\, dt + (\sin t\cos t + 1)\, dt.\\&= -t\sin t\cos t - \cos^2 t + t\,\Big|_0^{2\pi} = 2\pi.\end{aligned} \tag{19}$$

Next, we show the vector field is conservative. Here

$$P = yz, \quad Q = xz, \quad R = xy + 1, \tag{20}$$

so $\vec{F} = \langle yz,\ xz,\ xy+1, \rangle$ and

$$\begin{aligned}\nabla\times\vec{F} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ yz & xz & xy+1\end{vmatrix}\\ &= \langle 0, 0, 0\rangle\end{aligned} \tag{21}$$

so yes, it is. Next we find f. So

$$f_x = P = yz, \quad f_y = Q = xz, \quad f_z = R = xy + 1. \tag{22}$$

Integrating gives

$$\begin{aligned} f_x = yz &\quad\Rightarrow\quad f = xyz + A(y,z)\\ f_y = xz &\quad\Rightarrow\quad f = xyz + B(x,z)\\ f_z = xy+1 &\quad\Rightarrow\quad f = xyz + z + C(x,y)\end{aligned} \tag{23}$$

so

$$f = xyz + z \tag{24}$$

(again we neglect the c). Therefore,

$$\int_C yz\,dx + xz\,dy + (xy+1)\,dz = xyz + z\Big|_{(1,0,0)}^{(1,0,2\pi)} = 2\pi. \tag{25}$$

Line Integrals around closed curves

Let C be a smooth closed curve and $\vec{F}$ be a conservative vector field then

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \tag{26}$$

Example 3. Evaluate the following using the fundamental thm of line integrals.

$$\int_C 2xe^{x^2} \sin y dx + e^{x^2} \cos y\, dy \tag{27}$$

where C is CCW direction along circle $x^2 + y^2 = 1$.

Soln. First we show the vector field is conservative. Here

$$P = 2xe^{x^2} \sin y, \quad Q = e^{x^2} \cos y \text{ and } \quad Q_x = 2xe^{x^2} \cos y = P_y \tag{28}$$

so yes, it is. Next we find f.

$$f_x = P = 2xe^{x^2} \sin y, \quad f_y = Q = e^{x^2} \cos y \tag{29}$$

so

$$f = e^{x^2} \sin y \tag{30}$$

(we neglect the c). As for limits we pick a point on the circle, say $(1,0)$. We go around the circle CCW (although in this case direction doesn't matter) and arrive back at $(1,0)$ so

$$\int_C 2xe^{x^2}\sin y dx + e^{x^2}\cos y\, dy = e^{x^2}\sin y\Big|_{(1,0)}^{(1,0)} = 0. \tag{31}$$

Suppose we pick another point say $(0,1)$. We go around the circle CCW and arrive back at $(0,1)$ so

$$\begin{aligned}\int_C 2xe^{x^2}\sin y dx + e^{x^2}\cos y\, dy &= e^{x^2}\sin y\Big|_{(0,1)}^{(0,1)} \\ &= e^0\sin(1) - e^0\sin(1) = 0.\end{aligned} \tag{32}$$

Example 4. Evaluate

$$\oint_c 2y\, dx + x\, dy \tag{33}$$

where C is along circle $x^2 + y^2 = 1$ in the CCW direction. Unfortunately, the vector field is not conservative since

$$P = 2y, \quad Q = x \text{ and } \quad Q_x = 1 \neq P_y = 2. \tag{34}$$

However, there is a nice little theorem which relates the line integral over a vector field for closed curve to the region of the closed curve itself. It's called *Green's Theorem* and we'll cover this tomorrow.