Spring 2026 − Math 3331−Reduction of Order

The solution of linear nonhomgeneous ODEs

$$a(x)y'' + b(x)y' + c(x)y = f(x) \tag{1}$$

comprises of two parts

$$y = y_h + y_p \tag{2}$$

The y_h is the solution of the homogeneous problem

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{3}$$

and y_p some solution of the entire problem (called the particular solution).

Reduction of Order

This we've seen before. Given one solution of a second order ODE (say y_1) we seek the second solution using

$$y_2 = u(x)y_1. \tag{4}$$

Here, we use this same idea! The follow examples illustrate.

Example 1.

$$y'' - y = 2e^x \tag{5}$$

We first consider the homogeneous equation

$$y'' - y = 0 \tag{6}$$

The characteristic equation is

$$r^2 - -1 = 0 \tag{7}$$

which has roots $r = -1, 1$ so two independent solutions are $y = e^{-x}$ and $y = e^x$. We use one of these and let

$$y = e^x u \tag{8}$$

We calculate derivatives

$$\begin{aligned} y' &= e^x u' + e^x u \\ y'' &= e^x u'' + 2e^x u' + e^x u \end{aligned} \tag{9}$$

Now substitute into (5)

$$e^x u'' + 2e^x u' + e^x u - e^x u = 2e^x \tag{10}$$

and expanding gives

$$e^x \left(u'' + 2u'\right) = 2e^x \tag{11}$$

or

$$u'' + 2u' = 2 \tag{12}$$

If we let $u' = v$ and $u'' = v'$, then (12) becomes

$$v' + 2v = 2 \text{ (linear and separable)} \tag{13}$$

We solve giving

$$v = 1 + c_1 e^{-2x} \tag{14}$$

Now $u' = v$ so

$$u = x - \frac{1}{2}c_1 e^{-2x} + c_2 \tag{15}$$

Since $y = ue^x$ then we obtain

$$\begin{aligned} y &= \left(x - \frac{1}{2}c_1 e^{-2x} + c_2\right) e^x \\ &= xe^x - \frac{1}{2}c_1 e^{-x} + c_2 e^x \\ &= xe^x + c_1 e^{-x} + c_2 e^x \text{ (we adjusted the constant } c_1) \end{aligned} \tag{16}$$

Example 2.
Solve

$$y'' - 3y' + 2y = 1 + e^x \tag{17}$$

We first consider the homogeneous equation

$$y'' - 3y' + 2y = 0 \tag{18}$$

The characteristic equation is

$$r^2 - 3r + 2 = 0 \tag{19}$$

which has roots $r = 1, 2$ so two independent solutions are $y = e^x$ and $y = e^{2x}$. We use one of these and let

$$y = e^x u \tag{20}$$

We calculate derivatives

$$\begin{aligned} y' &= e^x u' + e^x u \\ y'' &= e^x u'' + 2e^x u' + e^x u \end{aligned} \tag{21}$$

Now substitute into (17)

$$e^x \left(u'' + 2u' + u\right) - 3e^x \left(u' + u\right) + 2e^x u = 1 + e^x \tag{22}$$

and expanding gives

$$e^x \left(u'' - u'\right) = 1 + e^x \tag{23}$$

noting that the term involving u is gone. If we let $u' = v$ and $u'' = v'$, then (23) becomes

$$v' - v = e^{-x} + 1 \tag{24}$$

This new ODE (eqn. (24)) is linear and first order. The integrating factor is $\mu = e^{-x}$ and so

$$\begin{aligned} e^{-x} \left(v' - v\right) &= e^{-x} \left(e^{-x} + 1\right) \\ \frac{d}{dx} \left(e^{-x} v\right) &= e^{-2x} + e^{-x} \\ e^{-x} v &= -\frac{1}{2} e^{-2x} - e^{-x} + c_1 \\ v &= -\frac{1}{2} e^{-x} - 1 + c_1 e^x \end{aligned} \tag{25}$$

Since $u' = v$ then

$$\begin{aligned} u' &= -\frac{1}{2} e^{-x} - 1 + c_1 e^x \\ u &= \frac{1}{2} e^{-x} - x + c_1 e^x + c_2 \end{aligned} \tag{26}$$

and the final solution

$$\begin{aligned} y &= e^x u \\ &= e^x \left(\frac{1}{2} e^{-x} - x + c_1 e^x + c_2\right) \\ &= \frac{1}{2} - x e^x + c_1 e^{2x} + c_2 e^x \end{aligned} \tag{27}$$

where we see that $y_p = \dfrac{1}{2} - x e^x$ and $y_h = c_1 e^{2x} + c_2 e^x$.

Example 3.

Solve

$$y'' + y = \tan x \tag{28}$$

We first consider the homogeneous equation

$$y'' + y = 0 \tag{29}$$

The characteristic equation is

$$r^2 + 1 = 0 \tag{30}$$

which has roots $r = -i, i$ so two independent solutions are $y = \cos x$ and $y = \sin x$. We use one of these and let

$$y = u \cos x. \tag{31}$$

We calculate derivatives

$$\begin{aligned} y' &= u' \cos x - u \sin x \\ y'' &= u'' \cos x - 2u' \sin x - u \cos x \end{aligned} \tag{32}$$

Now substitute into (28)

$$u'' \cos x - 2u' \sin x = \tan x \tag{33}$$

noting that the term involving u is gone. If we let $u' = v$ and $u'' = v'$, then (33) becomes

$$v' - 2\frac{\sin x}{\cos x} v = \frac{\sin x}{\cos^2 x} \tag{34}$$

This new ODE (eqn. (34)) is linear and first order. The integrating factor is $\mu = \cos^2 x$ and so

$$\begin{aligned} \cos^2 x \left(v' - 2\frac{\sin x}{\cos x} u' \right) &= \cos^2 x \frac{\sin x}{\cos^2 x} \\ \frac{d}{dx} \left(\cos^2 x v \right) &= \sin x \\ \cos^2 x v &= -\cos x + c_1 \\ v &= -\sec x + c_1 \sec^2 x \end{aligned} \tag{35}$$

Since $u' = v$ then

$$\begin{aligned} u' &= -\sec x + c_1 \sec^2 x \\ u &= -\ln|\sec x + \tan x| + c_1 \tan x + c_2 \end{aligned} \tag{36}$$

and the final solution

$$
\begin{aligned}
y &= \cos u \\
&= \cos x \left(-\ln|\sec x + \tan x| + c_1 \tan x + c_2\right) \\
&= -\cos x \ln|\sec x + \tan x| + c_1 \sin x + c_2 \cos x
\end{aligned} \tag{37}
$$

Here

$$
y_h = c_1 \sin x + c_2 \cos x, \quad y_p = -\cos x \ln|\sec x + \tan x|. \tag{38}
$$