# CAP 5993/CAP 4993
# Game Theory

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# Chicken

- The game of chicken models two drivers, both headed for a single-lane bridge from opposite directions. The first to swerve away yields the bridge to the other. If neither player swerves, the result is a costly deadlock in the middle of the bridge, or a potentially fatal head-on collision. It is presumed that the best thing for each driver is to stay straight while the other swerves (since the other is the "chicken" while a crash is avoided). Additionally, a crash is presumed to be the worst outcome for both players. This yields a situation where each player, in attempting to secure his best outcome, risks the worst.

# Chicken

| | Swerve | Straight |
|---|---|---|
| Swerve | Tie, Tie | Lose, Win |
| Straight | Win, Lose | Crash, Crash |

*Fig. 1: A payoff matrix of Chicken*

# Chicken

| | Swerve | Straight |
|---|---|---|
| Swerve | 0, 0 | -1, +1 |
| Straight | +1, -1 | -10, -10 |

*Fig. 2: Chicken with numerical payoffs*

- Game theory
  - Analyzing and computing solution concepts
- Modeling

- Random strategy:
  - *Increase cost/uncertainty to attackers*

**Adversary**

**Defender**

| | Target #1 | Target #2 |
|---|---|---|
| **Target #1** | 4, -3 | -1, 1 |
| **Target #2** | -5, 5 | 2, -1 |

# Rock-paper-scissors

| | **rock** | **paper** | **scissors** |
|---|---|---|---|
| Rock | 0,0 | -1, 1 | 1, -1 |
| Paper | 1,-1 | 0, 0 | -1,1 |
| Scissors | -1,1 | 1,-1 | 0,0 |

# Prisoner's dilemma

| | Prisoner B stays silent (*cooperates*) | Prisoner B betrays (*defects*) |
|---|---|---|
| **Prisoner A stays silent (*cooperates*)** | Each serves 1 year | Prisoner A: 3 years<br>Prisoner B: goes free |
| **Prisoner A betrays (*defects*)** | Prisoner A: goes free<br>Prisoner B: 3 years | Each serves 2 years |

| | C | D |
|---|---|---|
| C | 2<br>2 | 3<br>0 |
| D | 0<br>3 | 1<br>1 |

A payoff matrix of the standard dilemma of cooperation and defection

**Canonical PD payoff matrix**

| | Cooperate | Defect |
|---|---|---|
| **Cooperate** | *R*, *R* | *S*, *T* |
| **Defect** | *T*, *S* | *P*, *P* |

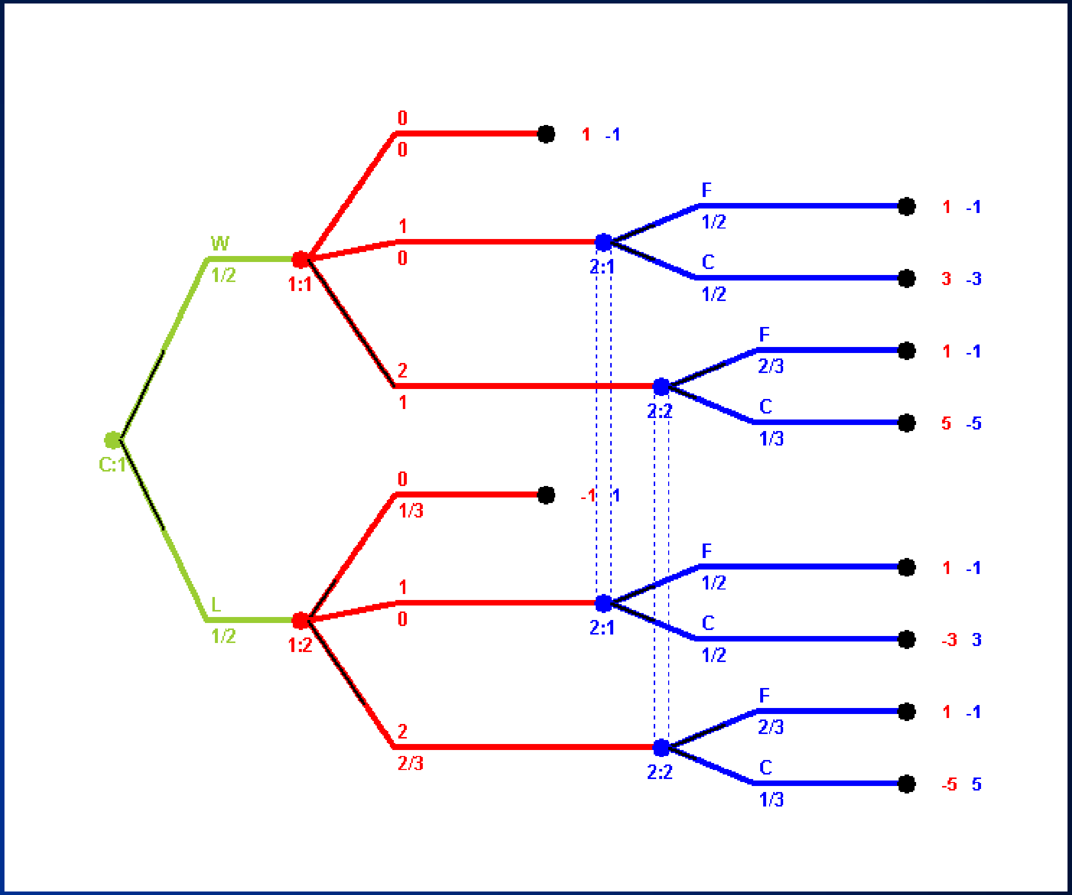
$$T > R > P > S$$

# Game Theory Explorer

- http://www.maths.lse.ac.uk/Personal/stengel/TEXTE/largeongte.pdf
- http://banach.lse.ac.uk/

# Gambit

- http://gambit.sourceforge.net/gambit15/gui.html



# Proofs

- Square root of 2 is irrational

# Proofs

- Infinitely many primes

# Proofs

- x is odd iff x+3 is even

# Proofs

- Sum of integers from 1 to n

# Chess

- A match has 3 possible outcomes:
  - Victory of White, if White captures the Black King
  - Victory for Black, if Black captures the White King
  - A draw, if:
    1. Black’s turn, but he has no possible legal moves available, and his King is not in check
    2. White’s turn, but he has no possible legal moves available, and his King is not in check
    3. Both players agree to declare a draw
    4. Board position precludes victory for both sides
    5. 50 consecutive turns have been played without a pawn having been moved and without the capture of any piece on the board, and the player whose turn it is requests that a draw be declared
    6. Same board position appears three times, and the player whose turn it is requests that a draw be declared

# Board positions

- Assume game is finite (number of possible turns is bounded)
- Let X denote set of all possible board positions.
  - Identity of each piece on board and board square on which it is located
- Board position does not provide full details on sequence of moves that led to it.

# Game situation

- Definition: a **game situation** (in the game of chess) is a finite sequence $(x_0,\ldots, x_K)$ of board positions in X satisfying
  – $x_0$ is the opening board position
  – For each even integer k, $0 <= k < K$, going from board position $x_k$ to $x_{k+1}$ can be accomplished by a single legal move on the part of White
  – For each odd integer k, $0<= k < K$, going from board position $x_k$ to $x_{k+1}$ can be accomplished by a single legal move on the part of Black
- Let H denote the set of all game situations.

# Strategy

- A **strategy** for White is a function $s_W$ that associates every game situation $(x_0,\ldots, x_K)$ in H, where K is odd, with a board position $x_{K+1}$ such that going from board position $x_K$ to $x_{K+1}$ can be accomplished by a single legal move on the part of White. Define a strategy $s_B$ for black analogously.

- Any pair of strategies $(s_W, s_B)$ determines an entire course of moves: in the opening move, White plays the move that leads to $x_1 = s_W(x_0)$. Black then plays the move leading to $x_2 = s_B(x_0, x_1)$, and so on.
- An entire course of moves is called a **play** of the game.
- Every play of the game ends in either a victory for White, a victory for Black, or a draw.
- A strategy for White is a **winning strategy** if it guarantees that White will win, no matter what strategy Black chooses.

# Winning strategy

- A strategy $s_W$ is a winning strategy for White if for every strategy $s_B$ of Black, the play of the game determined by the pair $(s_W, s_B)$ ends in victory for White. A strategy $s_W$ is a strategy guaranteeing at least a draw for White if for every strategy $s_B$ of Black, the play of the game determined by the pair $(s_W, s_B)$ ends in either a victory for White or a draw.

- If $s_W$ is a winning strategy for White, then any White player (or computer program) adoping that strategy is guaranteed to win, even if he faces the world's chess champion.

- Theorem: In chess, one and only one of the following must be true:
  i. White has a winning strategy
  ii. Black has a winning strategy
  iii. Each of the two players has a strategy guaranteeing at least a draw.
- Applies to ALL chess matches, not a particular match
- Theorem is significant because a priori it might have been the case that none of the alternatives was possible; one could have postulated that no player could ever have a strategy always guaranteeing a victory, or at least a draw.

# Checkers is Solved (Science '07)

- The game of checkers has roughly 500 billion billion possible positions ($5 \times 10^{20}$). The task of solving the game, determining the final result in a game with no mistakes made by either player, is daunting. Since 1989, almost continuously, dozens of computers have been working on solving checkers, applying state-of-the-art artificial intelligence techniques to the proving process. This paper announces that checkers is now solved: Perfect play ……

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# 2-player limit Hold’em poker is solved (Science 2015)

## Heads-up Limit Hold’em Poker is Solved

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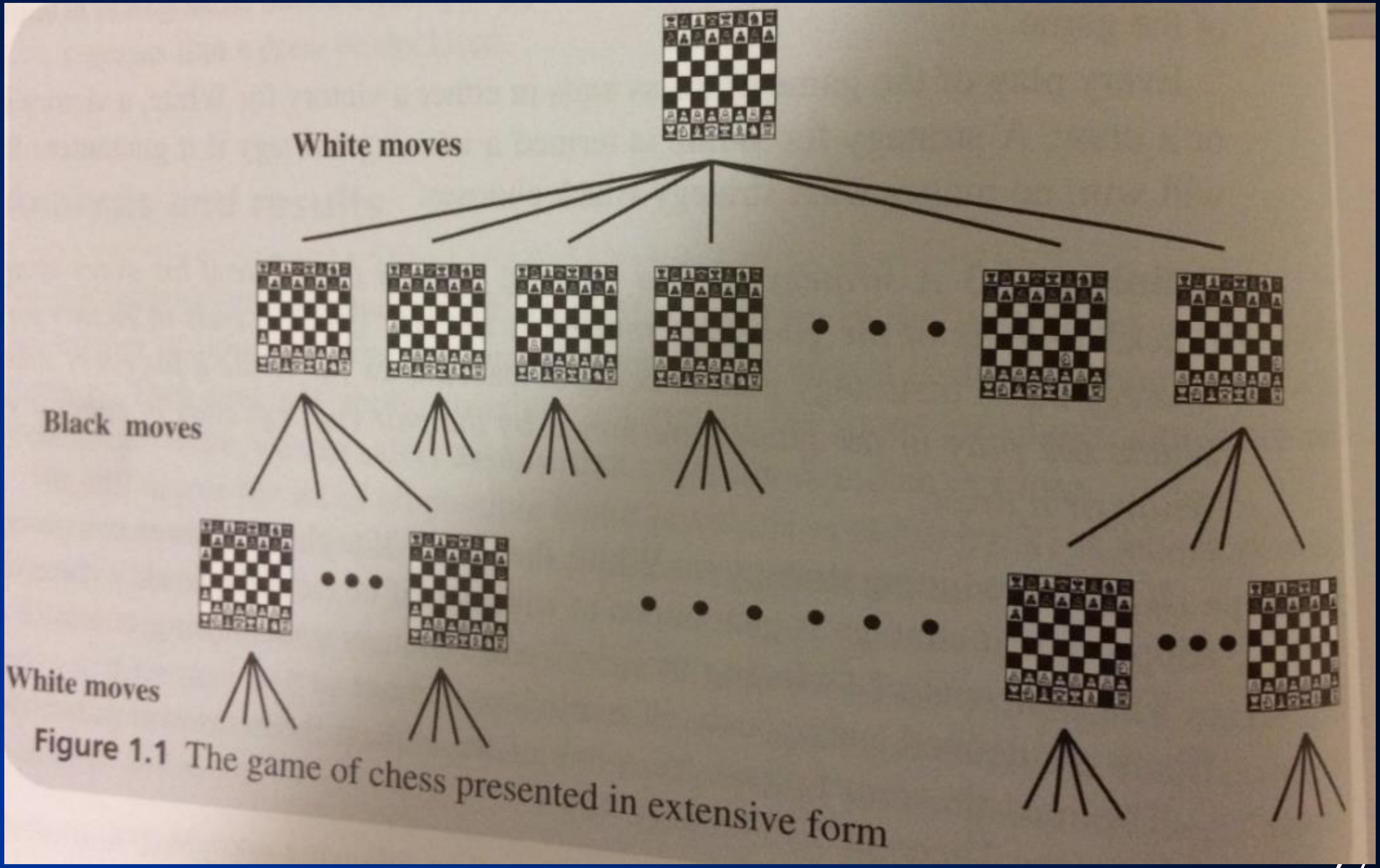
**Poker is a family of games that exhibit imperfect information, where players do not have full knowledge of past events. Whereas many perfect information games have been solved (e.g., Connect Four and checkers), no nontrivial imperfect information game played competitively by humans has previously been solved. Here, we announce that heads-up limit Texas hold’em is now essentially weakly solved. Furthermore, this computation formally proves the common wisdom that the dealer in the game holds a substantial advantage. This result was enabled by a new algorithm, CFR$^+$, which is capable of solving extensive-form games orders of magnitude larger than previously possible.**

# Heads-up Limit Hold 'em Poker is Solved

- Play against Cepheus here http://poker-play.srv.ualberta.ca/

# Proof Sketch of Theorem

- Set of game situations can be depicted by a tree



- Denote set of vertices of game tree by H.
- The *root vertex* is the opening game situation $x_0$, and for each vertex x, the set of *children vertices* of x are the set of game situations that can be reached from x in one legal move.
- Every vertex that can be reached from x by a sequence of moves is called a *descendant* of x
- Every *leaf* of the tree corresponds to a terminal game situation, in which a player has won or tie

- For each vertex x, consider the *subtree* beginning at x, which is the tree whose root is x that is obtained by removing all vertices that are not descendants of x. This subtree $\Gamma(x)$ corresponds to a game that is called the *subgame beginning at x*. Denote the number of vertices in $\Gamma(x)$ by $n_x$. The full game is $\Gamma(x_0)$.
- F denotes set of all subgames.

- Theorem: Every game in F satisfies one and only one of the following alternatives:
  i. White has a winning strategy
  ii. Black has a winning strategy
  iii. Each of the two players has a strategy guaranteeing at least a draw.
- Proof: Induction on $n_x$, number of vertices in subgame $\Gamma(x)$
- For $n_x=1$, x is terminal vertex.
- Suppose $n_x>1$. Assume by induction that at all vertices satisfying $n_y< n_x$, exactly one of the alternatives holds in $\Gamma(y)$.
- …

# Homework for next class

- 4.1-4.7 from Maschler textbook (strategic-form games)