Math 1497 Calculus II – Sample Test 1 Solutions

1. $\displaystyle\int \tan^{-1} x\, dx$

Integration by parts. If

$$\begin{aligned} u &= \tan^{-1} x, \quad v = x, \\ du &= \frac{dx}{1+x^2}, \quad dv = dx, \end{aligned}$$

then

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int \frac{x}{1+x^2}\, dx$$

and if $u = 1 + x^2$, then $du = 2xdx$ so

$$\begin{aligned} \int \tan^{-1} x\, dx &= x \tan^{-1} x - \frac{1}{2} \int \frac{du}{u}, \\ &= x \tan^{-1} x - \frac{1}{2} \ln|u| + c, \\ &= x \tan^{-1} x - \frac{1}{2} \ln|1+x^2| + c. \end{aligned}$$

2. $\displaystyle\int \frac{\ln x}{x^2}\, dx$

Integration by parts. If

$$\begin{aligned} u &= \ln x, \quad v = -\frac{1}{x}, \\ du &= \frac{dx}{x}, \quad dv = \frac{dx}{x^2}, \end{aligned}$$

then

$$\int \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + c.$$

3. $\displaystyle\int_0^1 x e^x\, dx$

Integration by parts. If

$$\begin{aligned} u &= x, \quad v = e^x, \\ du &= dx, \quad dv = e^x\, dx, \end{aligned}$$

then

$$\int_0^1 x e^x\, dx = x e^x\big|_0^1 - \int_0^1 e^x\, dx = x e^x\big|_0^1 - e^x\big|_0^1 = (e - 0) - (e - 1) = 1.$$

4. $\displaystyle\int \tan^3 x \sec^4 x\, dx$

Trig. integral. If $u = \tan x$ then $du = \sec^2 x\,dx$, then substituting gives

$$\int u^3(1+u^2)\,du = \int u^3 + u^5\,du = \frac{u^4}{4} + \frac{u^6}{6} + c = \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} + c.$$

5. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

This integral is improper

$$\lim_{b\to 1}\int_0^b \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1}\sin^{-1}x\Big|_0^b = \lim_{b\to 1}\sin^{-1}b = \frac{\pi}{2}.$$

6. $\displaystyle\int_1^5 \ln x\,dx$

Integration by parts. If

$$\begin{aligned} u &= \ln x, \quad v = x, \\ du &= \frac{dx}{x}, \quad dv = dx, \end{aligned}$$

then

$$\int_1^5 \ln x\,dx = x\ln x|_1^5 - \int_1^5 dx = x\ln x|_1^5 - x|_1^5 = 5\ln 5 - 4.$$

7. $\displaystyle\int \sec^3 x\,dx$

Integration by parts. If

$$\begin{aligned} u &= \sec x, \quad v = \tan x, \\ du &= \sec x\tan x\,dx, \quad dv = \sec^2 x\,dx, \end{aligned}$$

then

$$\begin{aligned} \int \sec^3 x\,dx &= \sec x\tan x - \int \sec x\tan^2 x\,dx \\ &= \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx \\ &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx \end{aligned}$$

so

$$2\int \sec^3 x\,dx = \sec x\tan x + \ln|\sec x + \tan x|,$$

so

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + c.$$

8. $\displaystyle\int_{-3}^3 \frac{dx}{4-x^2}$

This integral is improper because the integrand is undefined at $x = -2$ and $x = 2$. So we consider

$$\int_{-3}^{3} \frac{dx}{4-x^2} = \int_{-3}^{-2} \frac{dx}{4-x^2} + \int_{-2}^{2} \frac{dx}{4-x^2} + \int_{2}^{3} \frac{dx}{4-x^2}$$

and will show that one of the integrals (the last one) diverges.

$$\begin{aligned} \int_{2}^{3} \frac{dx}{4-x^2} &= \lim_{a\to 2} \int_{a}^{3} \frac{dx}{4-x^2}\,dx = \frac{1}{4} \lim_{a\to 2} \int_{a}^{3} \frac{1}{2+x} + \frac{1}{2-x}\,dx \\ &= \frac{1}{4} \lim_{a\to 2} \left(\ln|2+x| - \ln|2-x|\right)\Big|_{a}^{3} = \frac{1}{4} \lim_{a\to 2} \left(\ln 5 - \ln|2+a| + \ln|2-a|\right) = -\infty. \end{aligned}$$

Since one integral diverges, the entire integral diverges.

9. $\displaystyle\int \sin^2 x \cos^3 x\,dx$

Trig. integral. If we re-write the integral as

$$\int \sin^2 x \cos^2 x \cos x\,dx,$$

then if $u = \sin x$, then $du = \cos x\,dx$ and we have

$$\int \sin^2 x \cos^2 x \cos x\,dx = \int u^2(1-u^2)\,du = \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c.$$

10. $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$

This integral is improper because the infinite limit. So

$$\lim_{b\to\infty} \int_0^b xe^{-x^2}\,dx = -\frac{1}{2} \lim_{b\to\infty} e^{-x^2}\Big|_0^b = -\frac{1}{2} \lim_{b\to\infty} e^{-b^2} - 1 = -\frac{1}{2}(0-1) = \frac{1}{2}.$$

11. $\displaystyle\int \cos^3 x\,dx$

Trig. integral. If we re-write the integral as

$$\int \cos^2 x \cos x\,dx,$$

then if $u = \sin x$, then $du = \cos x\,dx$ and we have

$$\int \cos^2 x \cos x\,dx = \int (1-u^2)\,du = u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c.$$

12. $\displaystyle\int_{\pi/4}^{\pi/3} \tan^2 x \sec^2 x\,dx$

Trig. integral. If $u = \tan x$ then $du = \sec^2 x\,dx$. The limits becomes

$$x = \tfrac{\pi}{4} \quad\Rightarrow\quad u = \tan\tfrac{\pi}{4} = 1, \qquad x = \tfrac{\pi}{3} \quad\Rightarrow\quad u = \tan\tfrac{\pi}{3} = \sqrt{3}.$$

Therefore, we have

$$\int_1^{\sqrt{3}} u^2\,du = \frac{u^3}{3}\bigg|_1^{\sqrt{3}} = \frac{3\sqrt{3}}{3} - \frac{1}{3} = \sqrt{3} - \frac{1}{3}.$$

13. $\displaystyle\int \frac{dx}{(x^2+4)^{3/2}}$

Trig. substitution. If $x = 2\tan\theta$ then $dx = 2\sec^2\theta\,d\theta$ and substituting gives

$$\frac{2}{2^3}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta + c.$$

Since $\tan = \frac{x}{2}$, then $\sin\theta = \frac{x}{\sqrt{x^2+4}}$ and our answer is

$$\int \frac{dx}{(x^2+4)^{3/2}}\,dx = \frac{1}{4}\frac{x}{\sqrt{x^2+4}} + c.$$

14. $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{x\sqrt{x^2+1}}$

Trig. substitution. If $x = \tan\theta$ then $dx = \sec^2\theta\,d\theta$. The limits change and here

$$\begin{aligned} x = 1 \quad&\Rightarrow\quad \tan\theta = 1 \quad&\Rightarrow\quad \theta = \pi/4 \\ x = \sqrt{3} \quad&\Rightarrow\quad \tan\theta = \sqrt{3} \quad&\Rightarrow\quad \theta = \pi/3 \end{aligned}$$

and substituting gives

$$\int_{\pi/4}^{\pi/3} \frac{\sec^2\theta}{\tan\theta\sec\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \csc\theta\,d\theta = -\ln|\csc\theta + \cot\theta|\bigg|_{\pi/4}^{\pi/3} = -\ln\sqrt{3} + \ln(\sqrt{2}+1).$$

15. $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$

Trig. substitution. If $x = 2\sec\theta$ then $dx = 2\sec\theta\tan\theta\,d\theta$. Substituting gives

$$\int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + c = \ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + c,$$

noting that we have $\sec\theta = \frac{x}{2}$ and we deduce that $\tan\theta = \frac{\sqrt{x^2-4}}{2}$

16. $\displaystyle\int \frac{dx}{x^2+3x+2}$

Partial fractions

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

Multiplying by $(x+1)(x+2)$ gives

$$1 = A(x+2) + B(x+1).$$

so

$$\begin{array}{ll} 1) & 2A+B=1 \\ x) & \;\; A+B=0 \end{array}$$

so $A=1$ and $B=-1$. Thus,

$$\int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \ln|x+1| - \ln|x+2| + c.$$

17. $\displaystyle\int_0^{\frac{1}{2}} \frac{x^3\,dx}{\sqrt{1-x^2}}$

Trig. substitution. If $x=\sin\theta$, then $dx=\cos\theta\,d\theta$. The limits become

$$x=0 \;\Rightarrow\; \theta=0, \qquad x=\tfrac{1}{2} \;\Rightarrow\; \theta=\tfrac{\pi}{6}.$$

Substituting gives

$$\int_0^{\pi/6} \frac{\sin^3\theta \cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}} = \int_0^{\pi/6} \sin^3\theta\,d\theta = \int_0^{\pi/6} \sin^2\theta\sin\theta\,d\theta.$$

If we let $u=\cos\theta$ then $du=-\sin\theta\,d\theta$. The limits becomes

$$\theta=0 \;\Rightarrow\; u=1, \qquad \theta=\tfrac{\pi}{6} \;\Rightarrow\; u=\tfrac{\sqrt{3}}{2}.$$

So

$$-\int_1^{\frac{\sqrt{3}}{2}} (1-u^2)du = \int_{\frac{\sqrt{3}}{2}}^{1} (1-u^2)du = u - \frac{u^3}{3}\bigg|_{\frac{\sqrt{3}}{2}}^{1} = \left(1-\frac{1}{3}\right) - \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) = \frac{2}{3} - \frac{3\sqrt{3}}{8}.$$

18. $\displaystyle\int \frac{x^2\,dx}{(x^2+1)^{3/2}}$

Trig. substitution. If $x=\tan\theta$ then $dx=\sec^2\theta\,d\theta$ and substituting gives

$$\int \frac{\tan^2\theta\sec^2\theta}{\sec^3\theta}\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int \frac{(\sec^2\theta-1)}{\sec\theta}\,d\theta = \int \sec\theta - \cos\theta\,d\theta.$$

Each integrates separately to

$$\ln|\sec\theta + \tan\theta| - \sin\theta + c = \ln|\sqrt{x^2+1} + x| - \frac{x}{\sqrt{x^2+1}} + c$$

since $\tan\theta = x$ then $\sin\theta = \frac{x}{\sqrt{x^2+1}}$ and $\sec\theta = \sqrt{x^2+1}$.

19. $\displaystyle\int \frac{2x-1}{(x-1)(x-2)^2}\,dx$

Partial fractions

$$\frac{2x-1}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Multiplying by $(x-1)(x-2)^2$ gives

$$2x-1 = A(x-2)^2 + B(x-1)(x-2) + C(x-1).$$

so

$$\begin{array}{rc} 1) & 4A + 2B - C = -1 \\ x) & -4A - 3B + C = 2 \\ x^2) & A + B = 0 \end{array}$$

so $A = 1$, $B = -1$ and $C = 3$. Thus,

$$\int \left(\frac{1}{x-1} - \frac{1}{x-2} + \frac{3}{(x-2)^2} \right) dx = \ln|x-1| - \ln|x-2| - \frac{3}{x-2} + c.$$

20. $\displaystyle\int \sin^5 x \cos^4 x\,dx$

Trig. integral. If we re-write the integral as

$$\int \sin^4 x \cos^4 x \sin x\,dx,$$

then if $u = \cos x$, then $du = -\sin x\,dx$ and we have

$$\begin{aligned} \int \sin^4 x \cos^4 x \sin x\,dx &= -\int (1-u^2)^2 u^4\,du = -\int u^4 - 2u^6 + u^8\,du \\ &= -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + c = -\frac{\cos^5\theta}{5} + \frac{2\cos^7\theta}{7} - \frac{\cos^9\theta}{9} + c. \end{aligned}$$

21. $\displaystyle\int \frac{1}{x^3+x}\,dx$

Partial fractions

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiplying by $x(x^2+1)$ gives

$$1 = A(x^2+1) + (Bx+C)x.$$

so

$$\begin{array}{ll} 1) & A = 1 \\ x) & C = 0 \\ x^2) & A + B = 0 \end{array}$$

so $A = 1$, $B = -1$ and $C = 0$. Thus,

$$\int \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx = \ln|x| - \frac{1}{2} \ln|x^2+1| + c.$$

22. $\displaystyle\int \sin^2 x \cos^2 x \, dx$

Trig. integral. Since both powers are even, we use the double angles formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}.$$

Thus, the integral becomes

$$\int \frac{1-\cos 2x}{2} \frac{1+\cos 2x}{2} \, dx = \frac{1}{4} \int 1 - \cos^2 2x \, dx = \frac{1}{4} \int 1 - \underbrace{\left(\frac{1+\cos 4x}{2} \right)}_{again} dx.$$

So,

$$\frac{1}{8} \int 1 - \cos 4x \, dx = \frac{1}{8} \left(x - \frac{\sin 4x}{4} \right) + c.$$

23. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4} \, dx$

This integral is improper because the integrand is undefined at $x = 0$ and the infinite limits. So we consider

$$\int_{-\infty}^{0} \frac{dx}{x^4} + \int_{0}^{\infty} \frac{dx}{x^4}$$

and will show that one of the integrals (the last one) diverges.

$$\int_0^\infty \frac{dx}{x^4} \, dx = \lim_{a \to 0, b \to \infty} \int_a^b \frac{dx}{x^4} = \lim_{a \to 0, b \to \infty} -\frac{1}{3} \frac{1}{x^3} \bigg|_a^b = \lim_{a \to 0, b \to \infty} \frac{1}{3} \left(\frac{1}{a^3} - \frac{1}{b^3} \right) = \infty,$$

so the integral diverges.

24. $\displaystyle\int \frac{4x^2 - x + 7}{(x-1)(x^2+4)} \, dx$

Partial fractions

$$\frac{4x^2 - x + 7}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}.$$

Multiplying by $(x-1)(x^2+4)$ gives

$$4x^2 - x + 7 = A(x^2+4) + (Bx+C)(x-1).$$

so

$$\begin{aligned} 1) &\quad 4A - C = 7 \\ x) &\quad -B + C = -1 \\ x^2) &\quad A + B = 4 \end{aligned}$$

so $A = 2$, $B = 2$ and $C = 1$. Thus,

$$\int \left(\frac{2}{x-1} + \frac{2x+1}{x^2+4} \right) dx = 2 \ln|x-1| + \ln|x^2+4| + \frac{1}{2} \tan^{-1} \frac{x}{2} + c.$$

25. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x+1}}$

This integral is improper because the infinite limit. So we consider

$$\lim_{b\to\infty} \int_0^b \frac{dx}{\sqrt{x+1}} = \lim_{b\to\infty} 2\sqrt{x+1}\Big|_0^b = \lim_{b\to\infty} 2\sqrt{b+1} - 2 = \infty,$$

so the integral diverges.

26. $\displaystyle\int \tan^3 x \sec^3 x \, dx$

Trig. integral. We re-write the integral as

$$\int \tan^2 x \sec^2 x \sec x \tan x \, dx$$

If $u = \sec x$ then $du = \sec x \tan x \, dx$. Therefore, we have

$$\int (u^2 - 1)u^2 \, du = \int u^4 - u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c.$$

27. $\displaystyle\int_{-1}^2 \frac{dx}{x-2}$

This integral is improper because the integrand is undefined at $x = 2$. So we consider

$$\lim_{b\to 2} \int_{-1}^b \frac{dx}{x-2} = \lim_{b\to 2} \ln|x-2|\Big|_{-1}^b = \lim_{b\to 2} \ln|b-2| - \ln 3 = \infty,$$

since $\ln\, 0$ is undefined. Therefore, the integral diverges.

28. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}} \, dx$

Trig. substitution. If we let $x = 3\sec\theta$ then $dx = 3\sec\theta\tan\theta\, d\theta$. Therefore, we have

$$\int \frac{3\sec\theta\tan\theta\, d\theta}{3^2\sec^2\theta 3\tan\theta} = \frac{1}{9}\int \frac{d\theta}{\sec\theta} = \frac{1}{9}\int \cos\theta\, d\theta = \frac{1}{9}\sin\theta + c = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + c$$

since $\sec\theta = \frac{x}{3}$ and $\sin\theta = \frac{\sqrt{x^2-9}}{x}$.

29. $\displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

Trig. substitution. If we let $x = 3\tan\theta$ then $dx = 3\sec^2\theta\,d\theta$. Therefore, we have

$$\int \frac{3^3\tan^3\theta 3\sec^2\theta}{3\sec\theta}\,d\theta = 27\int \tan^3\theta\sec\theta d\theta = 27\int \tan^2\theta\sec\theta\tan\theta d\theta.$$

If $u = \sec\theta$, the $du = \sec\theta\tan\theta\,d\theta$, and our integral becomes

$$27\int (u^2-1)\,du = 27\left(\frac{u^3}{3} - u\right) + c = 9\sec^3\theta - 27\sec\theta + c.$$

and since $\tan\theta = \frac{x}{3}$ then $\sec\theta = \frac{\sqrt{x^2+9}}{3}$ and our final answer is

$$\frac{(x^2+9)^{3/2}}{3} - 9\sqrt{x^2+9} + c.$$

30. $\displaystyle\int_{\sqrt{2}}^{2} \frac{x^5 dx}{\sqrt{x^2-1}}$

Trig. substitution. If $x = \sec\theta$ then $dx = \sec\theta\tan\theta\,d\theta$. The limits become

$$x = \sqrt{2} \;\Rightarrow\; \theta = \tfrac{\pi}{4}, \qquad x = 2 \;\Rightarrow\; \theta = \tfrac{\pi}{3}.$$

Substituting gives

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^5\theta\sec\theta\tan\theta}{\tan\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^6\theta\,d\theta.$$

We could use the $\int \sec^n x\,dx$ reduction formula but I wil use the substitution, $u = \tan\theta$, so $du = \sec^2\theta\,d\theta$ while the limits become

$$\theta = \tfrac{\pi}{4} \;\Rightarrow\; u = 1, \qquad \theta = \tfrac{\pi}{3} \;\Rightarrow\; u = \sqrt{3}.$$

Thus, the integral becomes

$$\int_1^{\sqrt{3}} (1+u^2)^2\,du = \int_1^{\sqrt{3}} 1 + 2u^2 + u^4\,du = u + \frac{2u^3}{3} + \frac{u^5}{5}\bigg|_1^{\sqrt{3}} = \frac{24\sqrt{3}}{5} - \frac{28}{15}.$$