~~ODES~~

9/6/22

$$\dot{\bar{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{x} \qquad \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \qquad \lambda^2+1=0$$

$$\lambda = \pm i$$

$\lambda = i$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \bar{e} = 0 \qquad \bar{e} = \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$$

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$\bar{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\Phi = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

check

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix} = \Phi' \checkmark$$

$$\Phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{so} \quad \bar{x} = \Phi(t)\, \underbrace{\Phi^{-1}(0)}_{I}\, \bar{x}_0$$

$$\text{so} \quad \bar{x} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \bar{x}_0$$

$$\frac{dx}{dt} = ax \qquad x = c\,e^{at}$$

$$\frac{d\bar{x}}{dt} = A\bar{x} \qquad \bar{x} = \bar{c}\,e^{At} \quad \text{or} \quad e^{At}\bar{c} \quad \text{which}$$

previous ex.'s

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\bar{x} \qquad \bar{x} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{pmatrix}\bar{x}_0$$

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\bar{x} \qquad \bar{x} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\bar{x}_0$$

so $\bar{x} = e^{At}\,\bar{c}$

↑ is this $\Phi(t)\,\Phi^{-1}(0)$

e^{At} matrix exponential

$$\text{so} \quad e^{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}t} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{pmatrix} \quad ?$$

$$e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}t} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Def^n $e^A = I + A + \frac{A^2}{2!} + \cdots$

Ex1 $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2^2 \end{pmatrix}$

$A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2^3 \end{pmatrix} \cdots$ $A^n = \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}$

$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} t + \begin{pmatrix} 1 & 0 \\ 0 & 2^2 \end{pmatrix} \frac{t^2}{2!} - - - -$

$= \begin{pmatrix} 1 + t + \frac{t^2}{2!} \cdots & 0 \\ 0 & 1 + 2t + \frac{2^2 t^2}{2!} \cdots \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$

Ex2 $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$A^4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$e^{At} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&1\\0&1\end{pmatrix}t + \begin{pmatrix}1&2\\0&1\end{pmatrix}\frac{t^2}{2!} + \begin{pmatrix}1&3\\0&1\end{pmatrix}\frac{t^3}{3!}$$

$$= \begin{pmatrix} 1+t+\frac{t^2}{2!}\cdots & t+\frac{2t^2}{2!}+\frac{3t^3}{3!}\cdots \\ 0 & 1+t+\frac{t^2}{2!}\cdots \end{pmatrix}$$

what is $t+\frac{2t^2}{2!}+\frac{3t^3}{3!}\cdots$

$$= t+\frac{t^2}{1!}+\frac{t^3}{2!}+\frac{t^4}{3!}$$

$$= t\left(1+t+\frac{t^2}{2!}+\cdots\right) = te^t$$

$A = \begin{pmatrix}1&1\\-1&1\end{pmatrix}$ $A^2 = \begin{pmatrix}0&2\\-2&0\end{pmatrix}$ $A^3 = \begin{pmatrix}-2&2\\-2&-2\end{pmatrix}$ $A^4 = \begin{pmatrix}-4&0\\0&-4\end{pmatrix}$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&1\\-1&1\end{pmatrix}t + \begin{pmatrix}0&2\\-2&0\end{pmatrix}\frac{t^2}{2!} + \begin{pmatrix}-2&2\\-2&-2\end{pmatrix}\frac{t^3}{3!} + \begin{pmatrix}-4&0\\0&-4\end{pmatrix}\frac{t^4}{4!}$$

$1+t-\frac{2t^3}{3!}$ $\qquad t+\frac{2t^2}{2!}+\frac{2t^3}{3!}\cdots$

$-t-\frac{2t^2}{2!}-\frac{2t^3}{3!}$ $\qquad 1+t-\frac{2t^3}{3!}-\frac{4t^4}{4!}$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{t^3}{3!} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{t^4}{4!}$$

$$= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \cdots & t - \frac{t^3}{3!} \cdots \\ -t + \frac{t^3}{3!} \cdots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

so it's giving $\Phi(t)\, \Phi^{-1}(0)$

so how hard is it to use

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad A^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \cdots$$

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&1\\0&1\end{pmatrix}t + \begin{pmatrix}1&2\\0&1\end{pmatrix}\frac{t^2}{2!} + \begin{pmatrix}1&3\\0&1\end{pmatrix}\frac{t^3}{3!} + \begin{pmatrix}1&4\\0&1\end{pmatrix}\frac{t^4}{4!}$$

$$= \begin{pmatrix}1+t+\frac{t^2}{2!}+\frac{t^3}{3!} & t+\frac{2t^2}{2!}+\frac{3t^3}{3!}+\frac{4t^4}{4!}+\cdots \\ 0 & 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\cdots\end{pmatrix}$$

$$= \begin{pmatrix}e^t & te^t\\0 & e^t\end{pmatrix} \qquad *$$

Note $\dot{\bar{x}} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\bar{x}$ $\quad \dot{x} = x+y,\ \dot{y} = y$

so $y = c_2 e^t$ $\quad \dot{x} - x = c_2 e^{+t}$ $\quad \frac{d}{dt} e^{-t}x = c_2$

$e^{-t}x = c_2 t + c_1$ so $x = c_1 e^t + c_2 t e^t$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}e^t & te^t\\0 & e^t\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} \qquad \Phi(t) = \begin{pmatrix}e^t & te^t\\0 & e^t\end{pmatrix} \qquad \Phi(0) = 1$$

so

$$= \Phi(t)\,\Phi^{-1}(t)\begin{pmatrix}\vec{c}\end{pmatrix}$$

$$= \begin{pmatrix}e^t & te^t\\0 & e^t\end{pmatrix}\bar{c}$$ ← in (*)

$$\dot{\bar{x}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \bar{x}$$

Eigenvalues $\lambda = 1, 1, 1$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\lambda = 1$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \bar{e} = \bar{0}$$

$e_2 + e_3 = 0$

$e_3 = 0$

so $\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$(A - \lambda I)\bar{e}_2 = e_1$ $\qquad$ $(A - \lambda I)\bar{e}_3 = \bar{e}_2$

Further

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad A^3 = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \quad A^5 = \begin{pmatrix} 1 & 5 & 15 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 1 & n & ? \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

1 – 1
2 – 3
3 – 6
4 – 10
5 = 15

$? \; \frac{n(n+1)}{2}$