

Spring 2026 – Math 3331 – Undetermined Coefficients

The solution of linear nonhomogeneous ODEs

$$a(x)y'' + b(x)y' + c(x)y = f(x) \quad (1)$$

comprises of two parts

$$y = y_h + y_p \quad (2)$$

The y_h is the solution of the homogeneous problem

$$a(x)y'' + b(x)y' + c(x)y = 0 \quad (3)$$

and y_p some solution of the entire problem (called the particular solution).

Underdetermined Coefficients - Part 2

We first consider

$$y'' - 3y' + 2y = 2x + 1 \quad (4)$$

The homogeneous solution is

$$y_h = c_1e^x + c_2e^{2x} \quad (5)$$

Now for the particular solution. We try a solution of the form

$$y_p = Ax + B \quad (6)$$

where A and B are to be determined. We found that $A = 1$, $B = 2$ and so

$$y_p = x + 2 \quad (7)$$

and the general solution is

$$y = c_1e^x + c_2e^{2x} + x + 2. \quad (8)$$

So are there any limitations with this method? Well, it comes down to "good guessing". For example, consider

$$y'' - y' = 2? \quad (9)$$

The CE is

$$m^2 - m = 0 \quad (10)$$

so $m = 0$ and $m = 1$ and the complimentary solution is

$$y_c = c_1 + c_2e^x \quad (11)$$

Since the RHS of (9) is 2, a natural guess is

$$y_p = A \quad (12)$$

however, substitution into (9) gives

$$0 = 2? \quad (13)$$

so certainly there is a problem. Interesting enough this ODE is independent of y and we can use reduction of order to solve it. So if $y' = u$ then $y'' = u'$ and (9) becomes

$$u' - u = 2? \quad (\text{linear and separable}) \quad (14)$$

and can be solved giving

$$y = -2x + c_1 + c_2e^x \quad (15)$$

and one will notice the y_p we're looking for is

$$y_p = -2x. \quad (16)$$

If we consider

$$y'' - y' = e^x? \quad (17)$$

again RHS of (17) is e^x so we look for particular solution of the form

$$y_p = Ae^x \quad (18)$$

which gives

$$0 = e^x? \quad (19)$$

However, reduction of order (let $y' = u$) gives

$$y = 2xe^x + c_1 + c_2e^x \quad (20)$$

and $y_p = 2xe^x$. So a recap. We look at the right hand side (RHS) of the nonhomogeneous equation and guess. If the RHS has terms that belong to the complimentary solution, it necessary to bump up the solution with x 's.

Ex 1 Find the general solution to

$$y'' - 2y' + y = 2e^x? \quad (21)$$

The CE is

$$m^2 - 2m + 1 = 0 \quad (22)$$

so $m = 1$ and $m = 1$ and the complimentary solutions is

$$y_c = c_1 x e^x + c_2 e^x \quad (23)$$

The RHS of (21) is $2e^x$. As the complimentary solution has both e^x and $x e^x$ it necessary to bump twice and thus we look for a solution like

$$y_p = Ax^2 e^x \quad (24)$$

Substitution into (21) gives

$$Ax^2 e^x + 4Ax e^x + 2Ae^x - 2(Ax^2 e^x + 2Ax e^x) + Ax^2 e^x = 2e^x \quad (25)$$

and after cancellation

$$2Ae^x = 2e^x \quad (26)$$

so $A = 1$ and the general solution is

$$y = c_1 x e^x + c_2 e^x + x^2 e^x. \quad (27)$$

So are there limitations to this method. Well yes! It involves good guessing. So what about something like

$$y'' + y = \tan x \quad (28)$$

or

$$y'' + y = \frac{1}{x} \quad (29)$$