# Math 3331 − ODEs − Sample Final Solutions

1. $\dfrac{dy}{dx} = (\ln y - \ln x + 1)\,\dfrac{y}{x}$.

Solution: The equation is homogeneous. We re-write it as

$$\frac{dy}{dx} = \left(\ln\frac{y}{x} + 1\right)\frac{y}{x}.$$

If we let $y = xu$ so $\dfrac{dy}{dx} = x\dfrac{du}{dx} + u$ then

$$x\frac{du}{dx} + u = (\ln u + 1)u,$$

which separates

$$\frac{du}{u\ln u} = \frac{dx}{x} \quad \Rightarrow \quad \ln\ln u = \ln x + \ln c \quad \Rightarrow \quad u = e^{cx}.$$

Therefore,

$$\frac{y}{x} = e^{cx} \quad \text{or} \quad y = xe^{cx},$$

2. $x\,\dfrac{dy}{dx} + 2\,y = x^2y^2$.

Solution: The equation is Bernoulli, so we put in standard form

$$\begin{aligned} x\,\frac{dy}{dx} + 2\,y &= x^2y^2, \\ \frac{dy}{dx} + \frac{2}{x}\,y &= x\,y^2, \\ \frac{1}{y^2}\,\frac{dy}{dx} + \frac{2}{x}\,\frac{1}{y} &= x. \end{aligned}$$

We let $u = \dfrac{1}{y}$ so $\dfrac{du}{dx} = -\dfrac{1}{y^2}\frac{dy}{dx}$ and substituting gives

$$\begin{aligned} -\frac{du}{dx} + \frac{2}{x}u &= x, \\ \frac{du}{dx} - \frac{2}{x}u &= -x, \quad \left(\text{the integrating factor is } \mu = \frac{1}{x^2}\right) \\ \frac{d}{dx}\left(\frac{1}{x^2}u\right) &= -\frac{1}{x}. \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{1}{x^2}u &= c - \ln|x|, \\ u &= x^2(c - \ln|x|), \\ \frac{1}{y} &= x^2(c - \ln|x|), \\ y &= \frac{1}{x^2(c - \ln|x|)}. \end{aligned}$$

3. $\dfrac{dy}{dx} - y = 2e^x, \quad y(0) = 3.$

Solution: The equation is linear and already in standard form. The integrating factor is $\mu = e^{-x}$. Thus,

$$\begin{aligned} \frac{d}{dx}\left(e^{-x}y\right) &= 2, \\ e^{-x}y &= 2x + c, \quad \text{from the IC } c = 3, \\ e^{-x}y &= 2x + 3, \\ y &= (2x+3)e^x. \end{aligned}$$

4. $\dfrac{dy}{dx} = \dfrac{1 - 2xy^2}{1 + 2x^2y}, \quad y(1) = 1.$

Solution: The equation is exact. The alternate form is

$$(2xy^2 - 1)dx + (2x^2y + 1)dy = 0,$$

and it is an easy matter to verify

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x},$$

so $z$ exists such that

$$\begin{aligned} \frac{\partial z}{\partial x} &= M = 2xy^2 - 1 \quad \Rightarrow \quad z = x^2y^2 - x + A(y), \\ \frac{\partial z}{\partial y} &= N = 2x^2y + 1 \quad \Rightarrow \quad z = x^2y^2 + y + B(x), \end{aligned}$$

so we can choose $A$ and $B$ giving $z = x^2y^2 - x + y$ and the solution as $x^2y^2 - x + y = c$. Since $y(1) = 1$, this give $c = 1$ and the solution $x^2y^2 - x + y = 1$.

2. Solve the following

$(i)$ $\quad y'' - 5y' + 6y = 0, \quad y(0) = 1, \; y'(0) = 0$

Soln: The CE is $m^2 - 5m + 6 = 0$ so $(m-2)(m-3) = 0$ giving $m = 2, m = 3$. The solution is

$$y = c_1e^{2x} + c_2e^{3x}$$

The IC's gives $c_1 + c_2 = 1, 2c_1 + 3c_2 = 0$. Solving gives $c_1 = 3, c_2 = -2$ leading to the solution

$$y = 3e^{2x} - 2e^{3x}$$

$(ii)$ $\; y'' + 2y' + 10y = 0, \quad y(0) = -1, \; y'(0) = 4$

Soln: The CE is $m^2 + 2m + 10 = 0$ giving $m = -1 \pm 3i$. The solution is

$$y = c_1e^{-x}\cos 3x + c_2e^{-x}\sin 3x$$

The IC's gives $c_1 = -1, -c_1 + 3c_2 = 4$. Solving gives $c_1 = -1, c_2 = 1$ leading to the solution

$$y = -e^{-x}\cos 3x + e^{-x}\sin 3x$$

$(iii)$ $\; 4y'' - 4y' + y = 0, \quad y(0) = 0, \; y'(0) = 1$

Soln: The CE is $4m^2 - 4m + 1 = 0$ so $(2m-1)(2m-1) = 0$ giving $m = 1/2, m = 1/2$. The solution is

$$y = c_1e^{1/2x} + c_2xe^{1/2x}$$

The IC's gives $c_1 = 0, c_2 = 1$ leading to the solution

$$y = xe^{1/2x}$$

3. (i) Solve

$$\left(x^2 - 2x\right) y'' - \left(x^2 - 2\right) y' + 2(x-1)y = 0,$$

given that $y_1 = x^2$ is one solution.

Soln: Let $y = x^2 u$ so $y' = x^2 u' + 2xu$ and $y'' = x^2 u'' + 4xu' + 2u$. Substituting and simplifying gives

$$x(x-2)u'' - (x^2 - 4x + 6)u' = 0$$

Letting $u' = v$ so $u'' = v'$ gives

$$(x(x-2))v' - (x^2 - 4x + 6)v = 0.$$

Separating gives

$$\frac{dv}{v} = \frac{x^2 - 6x + 4}{x(x-2)},$$

which integrates to

$$v = \frac{(x-2)e^x}{x^3}.$$

Since $u' = v$ this integrates once more giving

$$u = \frac{e^x}{x^2}$$

and since $y = x^2 u$ we obtain the second solution $y = e^x$. Thus the general solution is

$$y = c_1 x^2 + c_2 e^x$$

5. (ii) Solve

$$xy'' - (x+1)y' + y = 0,$$

given that $y_1 = e^x$ is one solution.

Soln: Let $y = e^x u$ so $y' = e^x u' + e^x u$ and $y'' = e^x u'' + 2x^x u' + e^x u$. Substituting and simplifying gives

$$xu'' + (x+1)u' = 0$$

Letting $u' = v$ so $u'' = v'$ gives

$$xv' + (x-1)v = 0.$$

Separating gives

$$\frac{dv}{v} = \frac{1-x}{x},$$

which integrates to

$$v = xe^{-x}.$$

Since $u' = v$ this integrates once more giving

$$u = -(x+1)e^{-x}$$

and since $y = e^x u$ we obtain the second solution $y = -(x+1)$. Thus the general solution is

$$y = c_1 e^x + c_2(x+1)$$

noting that we absorbed the $-1$ into $c_2$.

4. Solve using any method (reduction of order, method of undetermined coefficients or variation of parameters)

$$(i) \qquad y'' - 6y' + 9y = e^{3x},$$

The homogeneous equation is

$$y'' - 6y' + 9y = 0$$

The characteristic equation for this is $m^2 - 6m + 9 = 0$ giving $m = 3, 3$. Thus, the complementary solution is

$$y = c_1 e^{3x} + c_2 x e^{3x}.$$

If we were to use variation of parameters

$$y = ue^{3x} + vxe^{3x}. \tag{1}$$

If we were to use reduction of order,

$$y = ue^{3x}. \tag{2}$$

We will do both.

*Variation of parameters*
Taking the first derivative, we obtain

$$y' = u'e^{3x} + 3ue^{3x} + v'xe^{3x} + (3x+1)ve^{3x},$$

from which we set

$$u'e^{3x} + v'xe^{3x} = 0, \tag{3}$$

leaving

$$y' = 3ue^{3x} + (3x+1)ve^{3x}. \tag{4}$$

Calculating one more derivative gives

$$y'' = 3u'e^{3x} + 9ue^{3x} + (3x+1)v'e^{3x} + (9x+6)ve^{3x}. \tag{5}$$

Substituting (1), (4) and (5) into the original ODE and canceling gives

$$\begin{aligned} 3u'e^{3x} + \cancel{9ue^{3x}} + (3x+1)v'e^{3x} &+ \cancel{(9x+6)ve^{3x}} \\ - \cancel{18ue^{3x}} \qquad\qquad &- \cancel{6(3x+1)ve^{3x}} \\ + \cancel{9ue^{3x}} \qquad\qquad &+ \cancel{9xve^{3x}} \qquad = e^{3x} \end{aligned} \tag{6}$$

or

$$3u'e^{3x} + (3x+1)v'e^{3x} = e^{3x}. \tag{7}$$

Equations (3) and (7) are two equations for $u'$ and $v'$ which we solve giving

$$u' = -x, \quad v' = 1.$$

Integrating each respectively gives

$$u = -\frac{1}{2}x^2; \;\; v = x$$

and from (1) we obtain the particular solution

$$y \;=\; \frac{1}{2}x^2e^{3x}$$

This then gives rise to the general solution

$$y = c_1e^{3x} + c_2xe^{3x} + \frac{1}{2}x^2e^{3x}.$$

*Reduction of Order*
Taking the first derivative of (2), we obtain

$$y' = u'e^{3x} + 3ue^{3x}, \tag{8}$$

and one more derivative

$$y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}. \tag{9}$$

Substituting (2), (8) and (9) into the original ODE and canceling gives

$$\begin{aligned} u''e^{3x} + \cancel{6u'e^{3x}} + \cancel{9ue^{3x}} & \\ - \cancel{6u'e^{3x}} - \cancel{18ue^{3x}} & \\ + \cancel{9ue^{3x}} &= e^{3x} \end{aligned} \tag{10}$$

or

$$u''e^{3x} = e^{3x}. \tag{11}$$

After we cancel the $e^{3x}$, we integrate twice giving $u = \frac{1}{2}x^2$ leading to the solution

$$y = \frac{1}{2}x^2 e^{3x}, \tag{12}$$

and the general solution as given before.
4. (ii) Solve using any method (reduction of order, method of undetermined coefficients or variation of parameters)

$$(ii) \qquad y'' - y' = 2x - 3x^2$$

Soln: The homogeneous equation is $y'' - y' = 0$ The associated CE is $m^2 - m = 0$ giving $m = 0, 1$. The two independent solutions are $y_1 = e^0 = 1$ and $y_2 = e^x$. Thus, the complementary solution is

$$y = c_1 + c_2 e^x$$

Here we will use the method of underdetermined coefficients. One would guess a particular solution of the form $y_p = Ax^2 + Bx + C$ but since $y = 1$ is a part of the complementary solution we need to bump the particular solution up by one. Thus, we try $y_p = Ax^3 + Bx^2 + Cx$. Substituting into the DE and comparing coefficients gives

$$\begin{aligned} x^2) \qquad -3A &= -3 \\ x) \quad 6A - 2B &= \phantom{-}2 \\ 1) \quad 2B - C &= \phantom{-}0 \end{aligned}$$

Solving gives $A = 1, B = 2$ and $C = 4$ giving $y_p = x^3 + 2x^2 + 4x$ and the general solution as

$$y = c_1 + c_2 e^x + x^3 + 2x^2 + 4x.$$

5(i)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \bar{x} \tag{13}$$

then the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 2$.

Case 1: $\lambda = -1$
In this case we have

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $2c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

so one solution is

$$\bar{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

Case 2: $\lambda = 2$
In this case we have

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $c_1 - c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

from which we obtain the other solution

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The general solution to (13) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

2(ii)
Consider

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \bar{x}, \quad \bar{x}(0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{14}$$

then the characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 9 = (\lambda-2)^2 = 0,$$

from which we obtain the eigenvalues $\lambda = 2$ and $\lambda = 2$ − repeated. As in problem 2(i) we find the eigenvector associated with this

Case 1: $\lambda = 2$

In this case we have

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so one solution is

$$\bar{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

For the second independent solution we seek a second solution of the form

$$\bar{x}_2 = \bar{u}te^{2t} + \bar{v}e^{2t}. \tag{15}$$

As shown in class, $\bar{u} = \bar{c}$ and $\vec{v}$ satisfies

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{16}$$

or $-v_1 - v_2 = 1$. Here, we'll choose

$$\bar{v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Therefore, the second solution is

$$\bar{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t}$$

and the general solution

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} t\, e^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t} \right],$$

Imposing the initial condition gives

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}.$$

This gives $c_1 - c_2 = 5$ and $-c_1 = -2$ so $c_1 = 2$ and $c_2 = -3$. The general solution then becomes

$$\bar{x} = 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} - 3 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} t\, e^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t} \right],$$

2(iii)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \bar{x}. \tag{17}$$

The characteristic equation is

$$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0.$$

Using the quadratic formula, we obtain $\lambda = 5 \pm 2i$ (so $\alpha = 5$ and $\beta = 2$). For the eigenvectors, we wish to solve

$$\begin{pmatrix} 6-(5+2i) & -1 \\ 5 & 4-(5+2i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\begin{pmatrix} 1-2i & -1 \\ 5 & -1-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which means solving

$$5v_1 - (1+2i)v_2 = 0.$$

One solution is

$$\bar{v} = \begin{pmatrix} 1+2i \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} i.$$

So here

$$\bar{A} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \vec{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

With $\alpha = 5$ and $\beta = 2$ gives

$$\vec{x}_1 = \left[ \begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] e^{5t}, \quad \vec{x}_2 = \left[ \begin{pmatrix} 1 \\ 5 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] e^{5t}.$$

The general solution is just a linear combination of these two

$$\vec{x} = c_1 \left[ \begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] e^{5t} + c_2 \left[ \begin{pmatrix} 1 \\ 5 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] e^{5t}.$$

6. Let $A = A(t)$ be the amount of salt at any time. Initially the tank contains pure water so $A(0) = 0$. The rate in is $r_i = 5$ gal/min and rate out $r_o = 10$ gal/min meaning the volume in the tank is decreasing so

$$V = V_0 + (r_i - r_o)t = 500 + (5-10)t = 500 - 5t$$

The change in salt at any time is given by

$$\frac{dA}{dt} = r_i c_i - r_o c_o$$

where $c_i$ and $c_o$ are concentrations in and out. Since we are given that $c_i = 2$ lb/gal and $c_o = A(t)/V(t)$ then we have

$$\begin{aligned}\frac{dA}{dt} &= 2 \cdot 5 - 10 \cdot \frac{A}{500 - 5t} \\ &= 10 - \frac{2A}{100 - t}\end{aligned}$$

This is linear so

$$\frac{dA}{dt} + \frac{2A}{100 - t} = 10$$

The integrating factor is $\mu = \exp\left(\int \frac{2}{100 - t} dt\right) = \exp\left(-2 \ln |100 - t|\right) = 1/(100 - t)^2$ so

$$\frac{d}{dt}\left(\frac{A}{(100 - t)^2}\right) = \frac{10}{(100 - t)^2}$$

Integrating gives

$$\frac{A}{(100 - t)^2} = \frac{10}{(100 - t)} + c$$

The initial condition $A(0) = 0$ gives $c = -1/10$ and finally giving the amount of salt at any time

$$A = 10(100 - t) - \frac{1}{10}(100 - t)^2.$$

When the tank is empty $V = 0$ which happens at $t = 100$ and $A(100) = 0$.

7. Let $P = P(t)$ be the population of rabbits. The differential equation is

$$\frac{dP}{dt} = kP(1000 - P)$$

Separating gives

$$\frac{dP}{P(1000 - P)} = kdt$$

or

$$\frac{1}{1000}\left(\frac{1}{P} + \frac{1}{1000 - P}\right) dP = kdt$$

and multiplying by 1000

$$\left(\frac{1}{P} + \frac{1}{1000 - P}\right) dP = 1000kdt$$

We can absorb the 1000 into the $k$. Integrating gives

$$\ln P + \ln(1000 - P) = kt + \ln c$$

or

$$\frac{P}{1000 - P} = ce^{kt} \tag{18}$$

Using the initial condition gives $P(0) = 100$ gives $c = 1/9$ and further $P(1) = 120$ gives $k = .204794$. Solving (18) for $P$ gives

$$P = \frac{1000e^{.204794t}}{e^{.204794t} + 9}.$$

To answer the questions $P(2) = 143.36$ so after two weeks there are 143 rabbits and the value of $t$ when $P = 900$ is $t = 21.46$ or roughly 21 and a half weeks.

8. Assuming Newton's law of cooling we have

$$\frac{dT}{dt} = k\left(T_\infty - T\right)$$

subject to $T(0) = 160$ and $T(20) = 150$. Here $T_\infty = 70$. Separating the DE gives

$$\frac{dT}{70 - T} = kdt$$

which we write as

$$\frac{dT}{T - 70} = -kdt$$

as $T$ is greater than the room temperature 70. Integrating gives

$$\ln T - 70 = -kt + \ln c$$

or

$$T = 70 + ce^{-kt}$$

Using $T(0) = 160$ gives $c = 90$ and using $T(20) = 150$ gives $k = .005882$. Thus, the temperature at any time is given by

$$T = 70 + 90e^{-.005882t}.$$