On the Melzak and Wilf Identities

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Abstract: Authors use the Stirling numbers of the second kind to show an identity of Wilf, which in turn implies a formula of Melzak.

Keywords: Wilf’s technique, Stirling numbers of the second kind, Melzak’s formula.

1. Introduction:

Wilf (Wilf, 2002) proved the relation:

\[ Q \equiv \sum_{k=m}^{n} P_j(k) A(k) x^k = P_j \left( x \frac{d}{dx} \right) \sum_{k=m}^{n} A(k) x^k, \]

where \( P_j(k) \) is a polynomial of degree \( j \). In Sec. 2 authors employ the Stirling numbers of the second kind \( S_n^{[q]} \) (Benjes, 1971; Quaintance & Gould, 2015; Barrera-Figueroa, López-Bonilla, & López-Vázquez, 2017) to show (1), and authors use it to motivate the following identity (Amdeberhan, De Angelis, & Moll, 2013):

\[ R_j \equiv \sum_{k=0}^{n} \mu_j(k) S_n^{[k]} = \sum_{k=0}^{n+j} S_n^{[k+j]}, \quad n, j \geq 0, \]

for the polynomials \( \mu_j(k) \) defined by:

\[ \mu_j(k) = k \mu_j(k) + \mu_j(k + 1), \quad \mu_0(k) = 1, \]

that is:

\[ \mu_1(k) = k + 1, \quad \mu_2(k) = k^2 + 2k + 2, \quad \mu_3(k) = k^3 + 3k^2 + 6k + 5, \ldots \]

Besides, (1) allows demonstrate the Melzak’s formula (Melzak, 1973):

\[ M \equiv \sum_{k=0}^{n} \binom{n}{k} k^p (n - k)^q u^k v^{n-k} = \left[ (x \frac{\partial}{\partial x})^p (y \frac{\partial}{\partial y})^q (u x + v y)^n \right]_{x=y=1}. \]

2. Wilf’s expression

Here authors use the Stirling numbers of the second kind to prove (1). First, let’s remember the property (Quaintance & Gould, 2015):

\[ k^r = \sum_{q=0}^{r} \binom{k}{q} q! S_r^{[q]}, \]

and the Grunert’s operational relation (Quaintance & Gould, 2015; Barrera-Figueroa, López-Bonilla, & López-Vázquez, 2017; Amdeberhan, De Angelis, & Moll, 2013; Melzak, 1973; Arakawa, Ibukiyama, & Kaneko, 2014) for the Euler operator (Stopple, 2003) \( x \frac{d}{dx} \):
\[
(\frac{d}{dx})^r f = \sum_{q=0}^r x^q S_r^q \frac{d^q}{dx^q} f ,
\] (7)
then:
\[
Q = \sum_{k=m}^n A(k) \sum_{r=0}^j a_{jr} k^r x^k , \quad P_j(k) = \sum_{r=0}^n a_{jr} k^r ,
\] (8)
but:
\[
k^r x^k = \sum_{q=0}^r \frac{k!}{(k-q)!} x^{k-q} S_r^q = \sum_{q=0}^r x^q S_r^q \frac{d^q}{dx^q} x^k = (\frac{d}{dx})^r x^k ,
\] (9)
Hence from (8):
\[
Q = \sum_{r=0}^n a_{jr} (\frac{d}{dx})^r \sum_{k=m}^n A(k) x^k = \text{eq. (1)}, \text{ q.e.d. ;}
\]
Besides, (1) for \( x = 1 \) gives the relation:
\[
\sum_{k=m}^n P_j(k) A(k) = [P_j (\frac{d}{dx}) \sum_{k=m}^n A(k) x^k]_{x=1} .
\] (10)
Now authors consider (2) for \( j = 1 \), thus from (1), (4) and (10):
\[
R_1 = [\left( (\frac{d}{dx}) + 1 \right) \sum_{k=0}^n x^k S_n^k]_{x=1} = [(1-x) \sum_{k=0}^n x^k S_n^k + \sum_{k=0}^{n+1} x^k S_n^k]_{x=1} = \sum_{k=0}^{n+1} S_n^k ,
\]
where was applied the identity (Quaintance & Gould, 2015):
\[
\sum_{k=0}^n x^k S_n^k = e^x \sum_{r=0}^\infty \frac{r^n}{r!} x^r .
\] (11)
Similarly, (2) for \( j = 2 \):
\[
R_2 = \{\left[ (\frac{d^2}{dx^2})^2 + 2 (\frac{d}{dx}) + 1 \right] \sum_{k=0}^n x^k S_n^k \}_{x=1} ,
\] (11)
\[
= [(1-x)(2-x) \sum_{k=0}^n x^k S_n^k + 2(1-x) \sum_{k=0}^{n+1} x^k S_n^k + \sum_{k=0}^{n+2} x^k S_n^k]_{x=1} = \sum_{k=0}^{n+2} S_n^k ,
\]
etc., hence with (1), (4), (10) and (11), authors can verify (2) for several values of \( j \).

The formula (1) allows prove (5), in fact:
\[
M^{(1), (10)} = [(x \frac{\partial}{\partial x})^p \Sigma_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) (n-k)^q (ux)^k v^{n-k}]_{x=1} = [(x \frac{\partial}{\partial x})^p \Sigma_{r=0}^n \left( \begin{array}{c} n \\ r \end{array} \right) r^q (ux)^{n-r} v^r]_{x=1} ,
\]
\[
(1), (10) = [(y \frac{\partial}{\partial y})^q \Sigma_{p=0}^n \left( \begin{array}{c} n \\ p \end{array} \right) (ux)^{n-r} (vy)^r]_{x=y=1} = \text{eq. (5)}, \text{ q.e.d.}
\]
For example, (5) for \( p = 0, q = n, v = -u = 1 \) implies:
\[
\sum_{k=0}^n (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) (n-k)^n = \left[ (y \frac{\partial}{\partial y})^n (y - 1)^n \right]_{y=1} = n! ,
\] (12)
which is a particular case of the identity (Amdeberhan, De Angelis, & Moll, 2013):
\[
\sum_{k=0}^n (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) (c + n - k)^{n-k} (n-k)^k = n! \sum_{j=0}^{c-i} \frac{c^j}{j!} ,
\] (13)
for \( c = 0 \).
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