On the Melzak and Wilf Identities

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Abstract: Authors use the Stirling numbers of the second kind to show an identity of Wilf, which in turn implies a formula of Melzak.

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1. Introduction:

Wilf (Wilf, 2002) proved the relation:

$$Q \equiv \sum_{k=m}^{n} P_j(k)\, A(k)\, x^k = P_j\left(x\frac{d}{dx}\right) \sum_{k=m}^{n} A(k)\, x^k, \tag{1}$$

where $P_j(k)$ is a polynomial of degree j. In Sec. 2 authors employ the Stirling numbers of the second kind $S_r^{[q]}$ (Benjes, 1971; Quaintance & Gould, 2015; Barrera-Figueroa, López-Bonilla, & López-Vázquez, 2017) to show (1), and authors use it to motivate the following identity (Amdeberhan, De Angelis, & Moll, 2013):

$$R_j \equiv \sum_{k=0}^{n} \mu_j(k)\ S_n^{[k]} = \sum_{k=0}^{n+j} S_{n+j}^{[k]}\ , \qquad n, j \geq 0, \tag{2}$$

for the polynomials $\mu_j(k)$ defined by:

$$\mu_j(k) = k\, \mu_j(k) + \mu_j(k+1), \qquad \mu_0(k) = 1, \tag{3}$$

that is:

$$\mu_1(k) = k+1, \qquad \mu_2(k) = k^2 + 2k + 2, \qquad \mu_3(k) = k^3 + 3k^2 + 6k + 5, \ldots \tag{4}$$

Besides, (1) allows demonstrate the Melzak's formula (Melzak, 1973):

$$M \equiv \sum_{k=0}^{n} \binom{n}{k}\ k^p\ (n-k)^q\ u^k\ v^{n-k} = [(\,x\frac{\partial}{\partial x})^p\ (\,y\frac{\partial}{\partial y})^q\ (u\,x + v\,y)^n]_{x=y=1}\,. \tag{5}$$

2. Wilf's expression

Here authors use the Stirling numbers of the second kind to prove (1). First, let's remember the property (Quaintance & Gould, 2015):

$$k^r = \sum_{q=0}^{r} \binom{k}{q}\ q!\ S_r^{[q]}\,, \tag{6}$$

and the Grunert's operational relation (Quaintance & Gould, 2015; Barrera-Figueroa, López-Bonilla, & López-Vázquez, 2017; Amdeberhan, De Angelis, & Moll, 2013; Melzak, 1973; Arakawa, Ibukiyama, & Kaneko, 2014) for the Euler operator (Stopple, 2003) $x\frac{d}{dx}$:

$$(x\frac{d}{dx})^r f = \sum_{q=0}^{r} x^q \, S_r^{[q]} \, \frac{d^q}{dx^q} f \, , \tag{7}$$

then:

$$Q = \sum_{k=m}^{n} A(k) \sum_{r=0}^{j} a_{jr} \, k^r \, x^k \, , \qquad P_j(k) = \sum_{r=0}^{j} a_{jr} \, k^r , \tag{8}$$

but:

$$k^r x^k \overset{(6)}{=} \sum_{q=0}^{r} \frac{k!}{(k-q)!} \, x^{k-q} \, x^q \, S_r^{[q]} = \sum_{q=0}^{r} x^q \, S_r^{[q]} \, \frac{d^q}{dx^q} x^k \overset{(7)}{=} (x\frac{d}{dx})^r \, x^k , \tag{9}$$

Hence from (8):

$$Q = \sum_{r=0}^{j} a_{jr} \, (x\frac{d}{dx})^r \sum_{k=m}^{n} A(k) \, x^k = \text{eq. (1), q.e.d. ;}$$

Besides, (1) for $x = 1$ gives the relation:

$$\sum_{k=m}^{n} P_j(k) \, A(k) = [P_j\left(x\frac{d}{dx}\right) \sum_{k=m}^{n} A(k) \, x^k]_{x=1} \, . \tag{10}$$

Now authors consider (2) for $j = 1$, thus from (1), (4) and (10):

$$R_1 = [\left(x\frac{d}{dx} + 1\right) \sum_{k=0}^{n} x^k \, S_n^{[k]}]_{x=1} = [\,(1-x) \sum_{k=0}^{n} x^k \, S_n^{[k]} + \sum_{k=0}^{n+1} x^k \, S_{n+1}^{[k]}]_{x=1} = \sum_{k=0}^{n+1} S_{n+1}^{[k]} ,$$

where was applied the identity (Quaintance & Gould, 2015):

$$\sum_{k=0}^{n} x^k \, S_n^{[k]} = e^{-x} \sum_{r=0}^{\infty} \frac{r^n}{r!} \, x^r \, . \tag{11}$$

Similarly, (2) for $j = 2$:

$$R_2 = \{\left[(x\frac{d}{dx})^2 + 2\left(x\frac{d}{dx} + 1\right)\right] \sum_{k=0}^{n} x^k \, S_n^{[k]}\}_{x=1} \, ,$$

$$\overset{(11)}{=} [(1-x)(2-x) \sum_{k=0}^{n} x^k \, S_n^{[k]} + 2(1-x) \sum_{k=0}^{n+1} x^k \, S_{n+1}^{[k]} + \sum_{k=0}^{n+2} x^k \, S_{n+2}^{[k]}]_{x=1} = \sum_{k=0}^{n+2} S_{n+2}^{[k]} \, ,$$

etc., hence with (1), (4), (10) and (11), authors can verify (2) for several values of j.

The formula (1) allows prove (5), in fact:

$$M \overset{(1),(10)}{=} [\left(x\frac{\partial}{\partial x}\right)^p \sum_{k=0}^{n} \binom{n}{k} (n-k)^q \, (ux)^k \, v^{n-k}]_{x=1} = [(x\frac{\partial}{\partial x})^p \sum_{r=0}^{n} \binom{n}{r} r^q \, (ux)^{n-r} v^r]_{x=1} \, ,$$

$$\overset{(1),(10)}{=} [\left(x\frac{\partial}{\partial x}\right)^p \left(y\frac{\partial}{\partial y}\right)^q \sum_{r=0}^{n} \binom{n}{r} (ux)^{n-r} (vy)^r \,]_{x=y=1} = \text{eq. (5), q.e.d.}$$

For example, (5) for $p = 0, q = n, v = -u = 1$ implies:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n = [\left(y\frac{d}{dy}\right)^n (y-1)^n \,]_{y=1} = n! \, , \tag{12}$$

which is a particular case of the identity (Amdeberhan, De Angelis, & Moll, 2013):

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (c+n-k)^{n-k} \, (n-k)^k = n! \sum_{j=0}^{n} \frac{c^j}{j!} \, , \tag{13}$$

for $c = 0$

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