Sample Final - Solutions

1. Find the unit tangent and unit normal vector for the following vector functions

$$\vec{r}(t) \ = < t, \frac{1}{2}t^2 >$$

Soln.

$$\begin{aligned}
\overrightarrow{r} &= \left\langle t, \frac{1}{2}t^2 \right\rangle \\
\overrightarrow{r}' &= \langle 1, t \rangle \\
\|\overrightarrow{r}'\| &= \sqrt{t^2+1}.
\end{aligned}$$

so

$$\overrightarrow{T} = \frac{\overrightarrow{r}'}{\|\overrightarrow{r}'\|} = \left\langle \frac{1}{\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}} \right\rangle$$

Further

$$\begin{aligned}
\overrightarrow{T}' &= \left\langle \frac{-t}{(t^2+1)^{3/2}}, \frac{1}{(t^2+1)^{3/2}} \right\rangle \\
\|\overrightarrow{T}'\| &= \frac{1}{t^2+1}.
\end{aligned}$$

so

$$\overrightarrow{N} = \frac{\overrightarrow{T}'}{\|\overrightarrow{T}'\|} = \left\langle \frac{-t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}} \right\rangle$$

2. Prove the limits either exist or do not exist. In the former case use the squeeze theorem.

$$(i) \lim_{(x,y)->(0,0)} \frac{x^2+2y^2}{x^2+y^2} \qquad (ii) \lim_{(x,y)->(0,0)} \frac{x^2y^4}{x^2+y^2}$$

Soln. 2 (i)

$$\begin{aligned}
&\text{Along} \quad y=0, \lim_{(x,y)->(0,0)} \frac{x^2+2y^2}{x^2+y^2} = \lim_{(x,y)->(0,0)} \frac{x^2}{x^2} = 1 \\
&\text{Along} \quad y=x, \lim_{(x,y)->(0,0)} \frac{x^2+2y^2}{x^2+y^2} = \lim_{(x,y)->(0,0)} \frac{3x^2}{2x^2} = \frac{3}{2}.
\end{aligned}$$

Since following different paths lead to different limits, the limit DNE.

Soln. 2 (ii) From the inequalities

$$\begin{aligned}
-\sqrt{x^2+y^2} \leq x \leq \sqrt{x^2+y^2} \\
-\sqrt{x^2+y^2} \leq y \leq \sqrt{x^2+y^2}
\end{aligned}$$

we have

$$-\left(x^2+y^2\right) \leq x^2 \leq \left(x^2+y^2\right)$$
$$-\left(x^2+y^2\right)^2 \leq y^4 \leq \left(x^2+y^2\right)^2$$

which gives

$$-\left(x^2+y^2\right)^3 \leq x^2y^4 \leq \left(x^2+y^2\right)^3.$$

Thus,

$$-\left(x^2+y^2\right)^2 \leq \frac{x^2y^4}{x^2+y^2} \leq \left(x^2+y^2\right)^2$$

and

$$-\lim_{(x,y)->(0,0)} \left(x^2+y^2\right)^2 \leq \lim_{(x,y)->(0,0)} \frac{x^2y^4}{x^2+y^2} \leq \lim_{(x,y)->(0,0)} \left(x^2+y^2\right)^2.$$

Since

$$\lim_{(x,y)->(0,0)} \left(x^2+y^2\right)^2 = 0$$

by the squeeze theorem

$$\lim_{(x,y)->(0,0)} \frac{x^2y^4}{x^2+y^2} = 0.$$

3. Find the equation of the tangent plane to the given surface at the specified point

$$x^2y + xz + yz^2 = 3, \quad P(1,2,-1)$$

Soln. If we define $F = x^2y + xz + yz^2 - 3$ then $F_x = 2xy + z$, $F_y = x^2 + z^2$ and $F_z = x + 2yz$. Evaluating these at the point P gives $F_x = 3$, $F_y = 2$ and $F_z = -3$. The equation of the tangent plane is thus $3(x-1) + 2(y-2) - 3(z+1) = 0$.

4. Find the directional derivative of $z = x^2 + 3xy + y^2$ at $(1,1)$ in the direction of $< -3, 4 >$.

Soln. The gradient is given by $\nabla z =< 2x + 3y, 3x + 2y >$ and at the point $(1,1)$ it becomes $\nabla z =< 5, 5 >$. The direction derivative is then given by

$$\nabla z \cdot \frac{\vec{u}}{\|\vec{u}\|} =< 5, 5 > \cdot \frac{< -3, 4 >}{5} = \frac{-15 + 20}{5} = 1.$$

5. Classify the critical points for

$$z = x^2y - x^2 + y^2 - 18y$$

Soln. The derivatives are

$$z_x = 2xy - 2x = 2x(y-1), \quad z_y = x^2 + 2y - 18.$$

Setting each of these to zero gives the following critical points: $(0, 9)$, $(-4, 1)$, and $(4, 1)$. The second derivatives are:

$$z_{xx} = 2(y-1), \quad z_{xy} = 2x, \quad z_{yy} = 2$$

giving $\Delta = z_{xx}z_{yy} - z_{xy}^2 = 4(y-1) - 4x^2$. We now test each critical point

$$\begin{array}{rlcc} (0,9) & \Delta = 32 > 0 & z_{yy} > 0 & \text{min} \\ (-4,1) & \Delta = -64 < 0 & & \text{saddle} \\ (4,1) & \Delta = -64 < 0 & & \text{saddle} \end{array}$$

6 (i). Find the volume bound by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$

Soln. The two surfaces intersect when $z = 0$ so $x^2 + y^2 = 1$. The volume is then obtained from the integral

$$\iint\limits_R \left(1 - x^2 - y^2\right) dA$$

As the region of integration is a circle of radius 1, we switch to polar coordinates giving

$$\int_0^{2\pi} \int_0^1 \left(1 - r^2\right) r dr d\theta = \frac{\pi}{2}$$

6 (ii). Find the volume inside the sphere $x^2 + y^2 + z^2 = 2$ and the cylinder $x^2 + y^2 = 1$

Soln. The surfaces intersect when $z^2 = 1$ or $z = \pm 1$. The volume is then obtained from the integral

$$\iint\limits_R 2\sqrt{2 - x^2 - y^2} dA$$

As the region of integration is a circle of radius 1, we switch to polar coordinates giving

$$\int_0^{2\pi} \int_0^1 2\sqrt{2 - r^2} r dr d\theta = \frac{8\sqrt{2} - 4}{3}\pi$$

6 (iii). Find the surface area of the plane $x + 2y + 3z = 6$ for $x, y, z \geq 0$.

Soln. The general formula is

$$\iint_R \sqrt{1 + z_x^2 + z_y^2}\, dA$$

Since $z = 2 - \frac{1}{3}x - \frac{2}{3}y$, the n$z_x = -1/3$ and $z_y = -2/3$ giving

$$\iint_R \sqrt{1 + \frac{1}{9} + \frac{4}{9}}\, dA = \frac{\sqrt{14}}{3} \iint_R dA$$

Thus

$$\frac{\sqrt{14}}{3} \int_0^3 \int_0^{6-2y} dx dy = 3\sqrt{14}.$$

7. Set of the triple integral $\iiint f(x, y, z)\, dV$ in both cylindrical and spherical coordinates for the volume inside the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 1$.

Soln - Cylindrical Eliminating z between the equations gives $x^2 + y^2 = 1$. This is the region of integration

$$\int_0^{2\pi} \int_0^1 \int_r^1 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Soln - Spherical From the picture we see that $\phi = 0 \to \pi/4$. Further, $\rho = 0 \to 1/\cos\phi$ and $\theta = 0 \to 2\pi$ so

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi,)\rho^2 \sin\phi d\rho\ d\phi\, d\theta$$

8. Is the following vector field conservative?

$$\vec{F} = < y^2 + 3yz, 2xy + 3xz, 3xy > .$$

Soln. Since $\nabla \times \vec{F} = 0$ then yes, the vector field is conservative. Thus f exists such that $\vec{F} = \vec{\nabla} f$ so

$$\begin{aligned} f_x &= y^2 + 3yz & \Rightarrow \quad f &= x\,y^2 + 3xyz + A(y,z) \\ f_y &= 2xy + 3xz & \Rightarrow \quad f &= x\,y^2 + 3xyz + B(x,z) \\ f_z &= 3xy & \Rightarrow \quad f &= 3xyz + C(x,y) \end{aligned}$$

Therefore we see that

$$f = x\,y^2 + 3xyz + c$$

and

$$\int_C \left(y^2 + 3yz\right)\, dx + (2xy + 3xz)\, dy + 3xy\, dz = \left. x\,y^2 + 3xyz \right|_{(0,0,0)}^{(1,2,3)} = 22.$$

9 (i). Evaluate the following line integral $\int_c 2xy\, dx + (x+1)\, dy$ where c is the counterclockwise direction around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and (0.1).

Soln. Here we have 4 separate curves which we denote by C_1, C_2, C_3 and C_4.

$C_1:$ Here $y = 0, dy = 0$ so $\displaystyle\int_{C_1} 0 = 0$

$C_2:$ Here $x = 1, dx = 0$ so $\displaystyle\int_0^1 2\, dy = 2$

$C_3:$ Here $y = 1, dy = 0$ so $\displaystyle\int_1^0 2x dx = -1$

$C_4:$ Here $x = 0, dx = 0$ so $\displaystyle\int_1^0 dy = -1$

Thus $\int_c 2xy\, dx + (x+1)\, dy = 0 + 2 - 1 - 1 = 0$.

9 (ii). Evaluate the following line integral $\int_c (x-y)\, dx + (x+y)\, dy$ where c is clockwise direction around the circle of radius 2.

Soln. Here we parameterize the curve by $x = 2\cos t,\ y = -2\sin t,\ 0 \le t \le 2\pi$. Note the -2 on the y term as we are going clockwise and not counterclockwise. So $dx = -2\sin t\, dt$ and $dy = -2\cos t\, dt$. Thus, the line integral becomes

$$\int_0^{2\pi} -(2\cos t + 2\sin t) 2\sin t\, dt - (2\cos t - 2\sin t) 2\cos t\, dt = -8\pi$$

10. Green's Theorem is

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Verify Green's Theorem where $\vec{F} = < 3x^2y, x^3 + x >$ where R is the region bound by the curves $y = x^2$ and $y = x$.

Soln. We have two separate curves which we denote by C_1 and C_2.

C_1 : Here $y = x^2, dy = 2x\,dx$ so $\displaystyle\int_0^1 3x^4dx + (x^3+x)2x\,dx = 5/3$

C_2 : Here $y = x, dy = dx$ so $\displaystyle\int_1^0 3x^3dx + (x^3+x)dx = -3/2$

so

$$\int_C 3x^2y\,dx + (x^3 + x)\,dy = 5/3 - 3/2 = 1/6.$$

For the second part, since $P = 3x^2y$ and $Q = x^3 + x$ then $Q_x - P_y = 3x^2 + 1 - 3x^2 = 1$ so

$$\iint_R (Q_x - P_y)\,dA = \int_0^1 \int_{x^2}^x 1dy\,dx = 1/6$$

11 (i). Evaluate $\displaystyle\iint_S x\,y\,dS$ where S is the surface of the plane $2x + y + z = 6$.

Soln. Since $z = 6 - 2x - y$ then $dS = \sqrt{1 + z_x^2 + z_y^2}\,dA = \sqrt{1+4+1}dA$ and thus

$$\int_0^3 \int_0^{6-2x} \sqrt{6}x\,y\,dy\,dx = 27\sqrt{6}/2$$

11 (ii). Evaluate $\displaystyle\iint_S (x+z)\,dS$ where S is the surface of the cylinder $y^2 + z^2 = 9$ bound between $x = 0$ and $x = 4$ in the first octant.

Soln. Without parameterization. We solve the surface for z so $z = \sqrt{9-y^2}$. Next $dS = \sqrt{1 + \frac{y^2}{9-y^2}}\,dA = \frac{3}{\sqrt{9-y^2}}\,dA$. As we have projected down, then the region of integration is in the xy plane $(0 \le x \le 4, 0 \le y \le 3)$. The surface integral becomes

$$\int_0^3 \int_0^4 \left(x + \sqrt{9-y^2}\right) \frac{3}{\sqrt{9-y^2}}\,dxdy = 3\int_0^3 \int_0^4 \left(\frac{x}{\sqrt{9-y^2}} + 1\right) dxdy = 12\pi + 36$$

With parameterization. Here, we'll parameterize the surface by $x = u$, $y = 3\cos v$ and $z = 3\sin v$, $0 \leq u \leq 3$ and $0 \leq v \leq \pi/2$. If we let $\vec{r} =< u, 3\cos v, 3\sin v >$ then $||\vec{r}_u \times \vec{r}_v|| = 3$ and we have

$$\int_0^4 \int_0^{\pi/2} (u + 3\sin v) 3\, dv\, du = 3(4\pi + 12)$$

This problem is really much easier with the parametric surface.

12. Verify the divergence theorem

$$\iint_S \vec{F} \cdot \vec{N}\, dS = \iiint_V \nabla \cdot \vec{F}\, dV$$

where $\vec{F} =< x + yz, y + xz, z + xy >$ and V is the volume of the tetrahedron bound by $x + y + z = 1$ and the planes $x = 0$, $y = 0$ and $z = 0$.

Soln. We will first deal with the surface integrals. There are 4 of them.

S_1: Bottom. Here $z = 0$ so $\vec{F} =< x, y, xy >$ and $\vec{N} =< 0, 0, -1 >$.
Thus $\vec{F} \cdot \vec{N} = -xy$ and $\int_0^1 \int_0^{1-x} -x\, y\, dy\, dx = -1/24$

S_2: Left. Here $y = 0$ so $\vec{F} =< x, xz, z >$ and $\vec{N} =< 0, -1, 0 >$.
Thus $\vec{F} \cdot \vec{N} = -xz$ and $\int_0^1 \int_0^{1-x} -x\, z\, dz\, dx = -1/24$

S_3: Back. Here $x = 0$ so $\vec{F} =< yz, y, z >$ and $\vec{N} =< -1, 0, 0 >$.
Thus $\vec{F} \cdot \vec{N} = -yz$ and $\int_0^1 \int_0^{1-y} -y\, z\, dz\, dy = -1/24$

S_4: Plane. Here $x + y + z = 1$ and $\vec{N} =< 1, 1, 1 > /\sqrt{3}$.

Thus, $\vec{F} \cdot \vec{N} = (1 + x + y - x^2 - xy - y^2)/\sqrt{3}$ and

$$\int_0^1 \int_0^{1-x} \left(1 + x + y - x^2 - xy - y^2\right)\, dy\, dx = 5/8$$

Therefore $\iint_S \vec{F} \cdot \vec{N} dS = -1/24 - 1/24 - 1/24 + 5/8 = 1/2.$

Second part. $\nabla \cdot \vec{F} = 3$ so $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 3\, dz\, dy\, dx = 1/2$. Verified!