Math 4315 - PDE's

So far we have been solving

$$a u_x + b u_y = cu + d$$

and in each case we have an arbitrary fct

Ex 1 $\quad u_x - u_y = 0 \qquad u = f(x+y)$

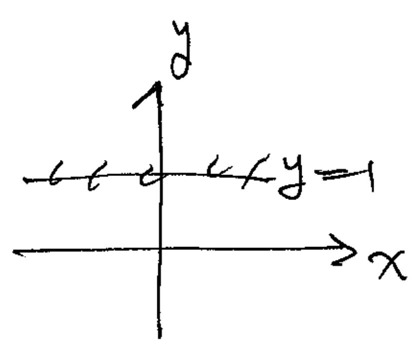
Ex 2 $\quad 2u_x + 3u_y = 1 \qquad u = \frac{x}{2} + f(3x - 2y)$

Ex 3 $\quad x u_x - y u_y = 2x \qquad u = 2x + f(xy)$

So how do we find the form of f?

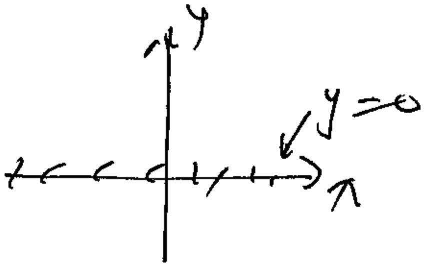
Initial/Boundary Condition

We now specify some more information in particular, information on a specific curve in the xy plane. For example



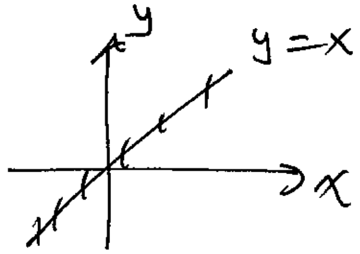
suppose we say that

$$u(x,1) = x$$



or $u(x,0) = 1$

or



$u(x,x) = 0$

on each of these curves $y=1$, $y=0$, $y=x$
we specify u ($u=x$, $u=1$, $u=0$)
It is these we use to find f

Ex 1 Solve $u_x - u_y = 0$, $u(x,0) = x^2$.

sol^n $u = f(x+y)$

Now BC $u(x,0) = x^2 \Rightarrow x^2 = f(x+0)$ so $f(x) = x^2$

sol^n $u = (x+y)^2$

Ex 2 Solve $2u_x + 3u_y = 1$ $\quad u(x,1) = 1+x$

sol^n $u = \frac{x}{2} + f(3x-2y)$

Now the B.C. $u(x,1) = 1+x$

$\Rightarrow x+1 = \frac{x}{2} + f(3x-2)$ ← How do we find f here?

$f(3x-2) = \frac{x}{2} + 1$

let $\lambda = 3x-2$ so $x = \frac{\lambda+2}{3}$

$f(\lambda) = \frac{1}{2}\left(\frac{\lambda+2}{3}\right) + 1$

$= \frac{\lambda}{6} + \frac{4}{3}$

so $u = \frac{x}{2} + \frac{1}{6}(3x-2y) + \frac{4}{3}$

$= \frac{x}{2} + \frac{x}{2} - \frac{y}{3} + \frac{4}{3}$

$u = x - y/3 + 4/3$

check $u_x = 1, \; u_y = -\frac{1}{3}$

$2u_x + 3u_y = 2 - \frac{3}{3} = 2-1 = 1$ ✓ PDE

$u(x,1) = x - \frac{1}{3} + \frac{4}{3} = x+1$ ✓

so the PDE & BC checks ✓

Ex Solve

$$x u_x + 2y u_y = u, \quad u(x,x) = 1$$

CE

$$x_s = x \Rightarrow x = a(r) e^{s}$$

$$y_s = 2y \Rightarrow y = b(r) e^{2s}$$

$$u_s = u \Rightarrow u = c(r) e^{s}$$

① $\dfrac{x^2}{y} = \dfrac{a^2(r) e^{2s}}{b(r) e^{2s}} = \dfrac{a^2}{b} = A(r)$

② $\dfrac{u}{x} = \dfrac{c(r) e^{s}}{a(r) e^{s}} = B(r)$

so $\dfrac{u}{x} = B\left(A^{-1}\left(\dfrac{x^2}{y}\right)\right) \Rightarrow u = x f\left(\dfrac{x^2}{y}\right)$

B.C. $u(x,x) = 1 \Rightarrow 1 = x f\left(\dfrac{x^2}{x}\right) \Rightarrow 1 = x f(x)$

$$f(x) = \frac{1}{x}$$

$$u = x \cdot \frac{1}{\frac{x^2}{y}} = x \frac{y}{x^2} = \frac{y}{x}$$

$u_x = -y/x^2, \quad u_y = \dfrac{1}{x}$

Sol^n $u = \dfrac{y}{x}$ B.C. $u(x,x) = \dfrac{x}{x} = 1$ ✓

$$x u_x + 2y u_y = \frac{-y}{x} + \frac{2y}{x} = y/x = u \checkmark$$