Math 4315 – PDE's

Now we are ready to solve the big 3 PDEs the heat eq^n, Laplace's eq^n & wave eq^n

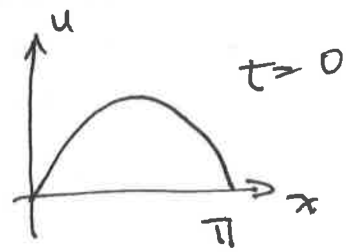
They are
$$u_t = u_{xx}$$
$$u_{xx} + u_{yy} = 0$$
$$u_{tt} = c^2 u_{xx} \qquad c - \text{constant}$$

First up – the heat eq^n

Solve $\quad u_t = u_{xx} \quad 0 < x < \pi$

subject to $\quad u(0,t) = u(\pi, t) = 0$

$\qquad\qquad u(x,0) = \sin x$



Separation of Variables

We assume a sol^n of the form

$$u(x,t) = X(x)\,T(t)$$

B.C. $\quad u(0,t) = 0 \Rightarrow X(0)\,T(t) = 0 \Rightarrow X(0) = 0$

$\qquad u(\pi,t) = 0 \Rightarrow X(\pi)\,T(t) = 0 \Rightarrow X(\pi) = 0$

The IC we consider later

Now sub into PDE

$$u_t = X T', \quad u_{xx} = X'' T$$

so $X T' = X'' T$

Now sep. $\dfrac{T'}{T} = \underline{\dfrac{X''}{X} = \lambda}$ same const

so $X'' = \lambda X$ subject to $X(0) = 0$
$X(\pi) = 0$

3 cases for λ

(i) $\lambda > 0$ set $\lambda = \omega^2$

ii) $\lambda = 0$

iii) $\lambda < 0$ set $\lambda = -\omega^2$

(i) $\underline{\lambda > 0}$

so $X'' = \omega^2 X \Rightarrow X'' - \omega^2 X = 0$

$$X = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

$$\left.\begin{aligned} X(0) = 0 &\Rightarrow c_1 + c_2 = 0 \\ X(\pi) = 0 & \quad c_1 e^{\pi\omega} + c_2 e^{-\pi\omega} = 0 \end{aligned}\right\} \quad c_1 = c_2 = 0$$

so $X \equiv 0$ so $u = XT \equiv 0$

can't be because $u(x,0) = \sin x$.

ii) $\underline{\lambda = 0}$

$X'' = 0 \Rightarrow X = c_1 x + c_2$

$X(0) = 0 \Rightarrow c_2 = 0$

$X(\pi) = 0 \Rightarrow c_1\pi + c_2 = 0$ $\Big\}$ $c_1 = c_2 = 0$

can't be

iii) $\underline{\lambda < 0}$

$X'' = -\omega^2 X \Rightarrow X'' + \omega^2 X = 0$

so $X = c_1 \sin\omega x + c_2 \cos\omega x$

$X(0) = 0 \Rightarrow c_2 = 0$ $\because$ $\sin 0 = 0$, $\cos 0 = 1$

$X(\pi) = 0 \Rightarrow c_1 \sin\omega\pi = 0$ $\omega = 0, \pm 1, \pm 2 \dots$

$\omega = n$ integer

so $X = c_1 \sin nx$

$\frac{T'}{T} = -\omega^2 = -n^2$ $\frac{dT}{T} = -n^2 dt$

$\ln T = -n^2 t + \ln c_3$ $T = c_3 e^{-n^2 t}$

Now we combine

$$u = XT = c_1 c_3 e^{-n^2 t} \sin nx$$

call $c_1 c_3 = b_n$ a constant but can change when n changes

so $u = b_n e^{-n^2 t} \sin n\pi$

Now the I.C.

$$u(x,0) = \sin x$$

so $u(x,0) = b_n e^{0} \sin n\pi$ compare

so $b_n = 1$ $n = 1$

so the sol^n is

$$\boxed{u(x,t) = e^{-t} \sin x}$$

Ex 2 Solve $u_t = u_{xx}$ $0 < x < \pi$

subject to $u(0,t) = u(\pi,t) = 0$

$u(x,0) = 4 \sin 3x$

Do we need to repeat the whole process – no!

why? The PDE & boundary condition are the same just the initial condition (IC) changes

so we still have

$$u(x,t) = b_n e^{-n^2 t} \sin nx$$

Compare with IC $u(x,0) = 4 \sin 3x$

gives $b_n = 4$ $n = 3$

so the sol^n is

$$u(x,t) = 4 e^{-9t} \sin 3x$$

Ex3

Solve $u_t = u_{xx}$ $0 < x < \pi$

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,0) = \sin x + 4 \sin 3x$$

Sol^n $u(x,t) = b_n e^{-n^2 t} \sin nx$

How do we pick n?

Superposition Principle

If u_1 & u_2 are sol^n of the heat eq^n

so is $u = c_1 u_1 + c_2 u_2$

so here we would add the 2 sol^n's giving

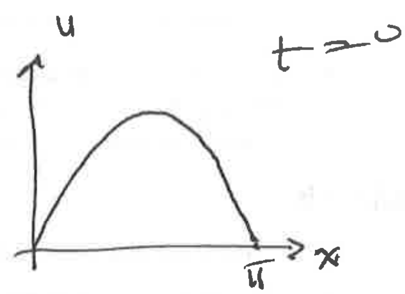
$$u = e^{-t}\sin x + 4e^{-9t}\sin 3x$$

Solve

$$u_t = u_{xx} \quad 0 < x < \pi$$

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,0) = \pi x - x^2$$



look at

$$u = \frac{8}{\pi}e^{-t}\sin x$$

$$u = \frac{8}{\pi}e^{-t}\sin x + \frac{8}{27\pi}e^{-9t}\sin 3x$$

$$u = \frac{8}{\pi}e^{-t}\sin x + \frac{8}{27\pi}e^{-9t}\sin 3x + \frac{8}{125\pi}e^{-25t}\sin 5x$$

How did I know how to choose these?