Math 4315 PDEs

1st order PDEs w/ BC.

$u(x,0) = x^2 + x$

Ex $\quad x u_x + (x-y) u_y = x \qquad$ ~~$u(1,x) = 2x$~~

$x_s = x \Rightarrow x = a(r) e^s$

$y_s = x - y \qquad y_s + y = x = a(r) e^s \qquad \frac{\partial}{\partial s} e^s y = a(r) e^{2s}$

$u_s = x \qquad e^s y = \frac{a(r) e^{2s}}{2} + b(r)$

so $y = \frac{a(r)}{2} e^s + b(r) e^{-s}$

$u_s = a(r) \cdot e^s \qquad u = a(r) e^s + c(r)$

so for $\quad x = a(r) e^s \qquad y - \frac{x}{2} = b(r) e^{-s}$

$\left. \begin{array}{l} \\ y = \frac{a(r)}{2} e^s + b(r) e^{-s} \end{array} \right] \qquad x\left(y - \frac{x}{2}\right) = ab = d(r)$

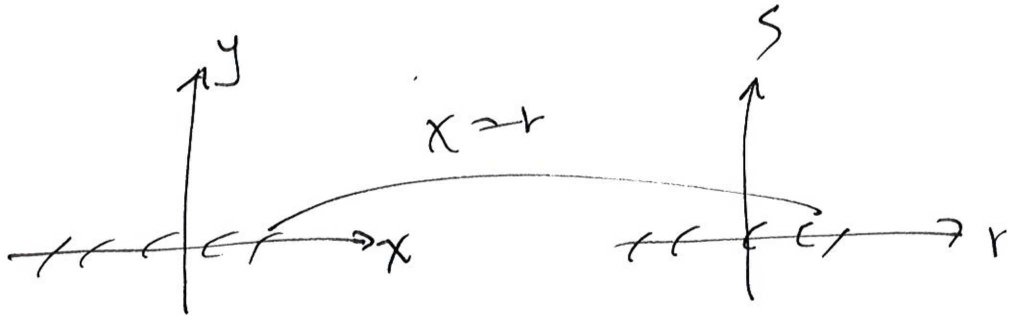
$u = a(r) e^s + c(r) \qquad u - x = c(r)$

$u = x + f\left(x\left(y - \frac{x}{2}\right)\right) \qquad$ BC. $\cancel{x} + x^2 = \cancel{x} + f\left(-\frac{x^2}{2}\right)$

$\lambda = -\frac{x^2}{2} \qquad f(\lambda) = -2\lambda \qquad u = x - 2\left[xy - \frac{x^2}{2}\right]$

$u = x + x^2 - 2xy$

Bring BC in early



$s=0$

$x_s = x$ $\quad x = r$

$y_s = x - y$ $\quad y = 0$

$u = r + r^2$

$u_s = x$

(1) $x_s = x$ $\quad x = a(r)e^s$ $\quad s=0 \; x=r$ $\quad a = r$ $\quad \boxed{x = re^s}$

(2) $y_s = x - y$ $\quad y_s + y = re^s$ $\quad \frac{\partial}{\partial s} e^s y = re^{2s}$

$e^s y = \frac{re^{2s}}{2} + b(r)$ $\quad y=0 \; s=0$ $\quad b = -r/2$

$e^s y = \frac{re^{2s}}{2} - \frac{r}{2}$ so $\boxed{y = \frac{re^s}{2} - \frac{re^{-s}}{2}}$

$u_s = x = re^s$ $\quad u = re^s + c(r)$ $\quad s=0 \; u = r + r^2$ $\quad c(r) = r^2$

$\boxed{u = re^s + r^2}$

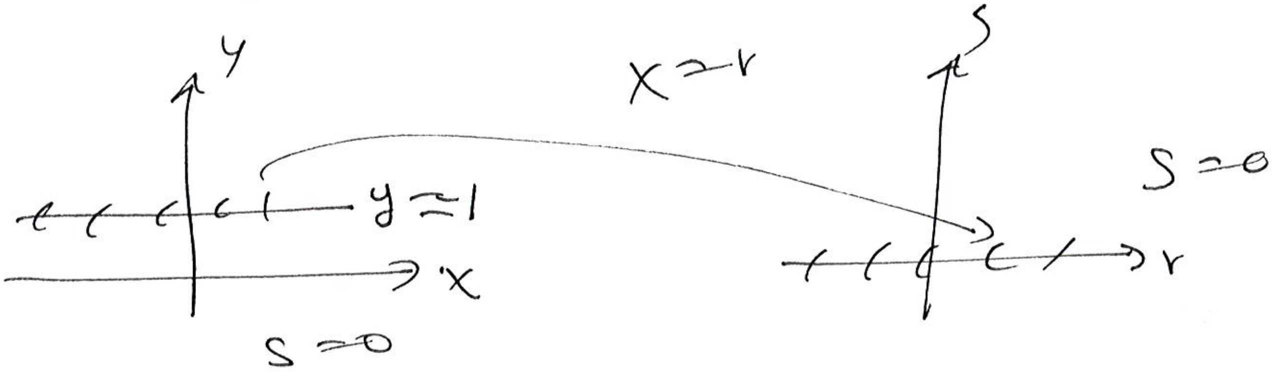
$y - \frac{x}{2} = -\frac{re^{-s}}{2}$, $x = re^s$

$x\left(y - \frac{x}{2}\right) = -\frac{r^2}{2}$ $\quad r^2 = 2\left(\frac{x}{2} - y\right)x$

$u = x + 2x\left(\frac{x}{2} - y\right)$

$= x + x^2 - 2xy$

BC early



$x_s = 1 \quad x = r$

$y_s = 2y \quad y = 1$

$u_s = u^2 \quad u = \frac{1}{r}$

$x = s + a(r) \quad s=0 \;\; x=r \quad a=r$ so $\boxed{x = s + r}$

$y_s = 2y \quad y = b(r)e^{2s} \quad s=0 \;\; y=1$ so $b=1$

$$\boxed{y = e^{2s}}$$

$u_s = u^2 \quad -\frac{1}{u} = s + c(r) \quad s=0 \;\; u = \frac{1}{r}$ so $c = -r$

$-\frac{1}{u} = s - r$ so $\ln y = 2s$

$= \cancel{2s}$

$s = \frac{1}{2}\ln y \quad r = x - \frac{1}{2}\ln y$

$-\frac{1}{u} = \frac{1}{2}\ln y - x + \frac{1}{2}\ln y = \ln y - x$

$u = \frac{1}{x - \ln y}$

ex 2 $\quad u_x + 2y\, u_y = u^2 \qquad u(x,1) = \frac{1}{x}$

if $u_s = u_x x_s + u_y y_s$

$$\left.\begin{array}{lll} x_s = 1 & \Rightarrow & x = s + a(r) \\ y_s = 2y & \rightarrow & y = b(r)\, e^{2s} \end{array}\right] \qquad \ln y = \ln b(r) + 2s$$

$$u_s = u^2 \Rightarrow \frac{-1}{u} = s + c(r) \qquad 2x - \ln y = 2a(r) - \ln b(r) = d(r)$$

$$x + \frac{1}{u} = a(r) - c(r) = e(r)$$

so $\quad x + \frac{1}{u} = f(2x - \ln y)$

BC $\quad x + x = f(2x)$

$$f(2x) = 2x$$

$$x + \frac{1}{u} = 2x - \ln y \Rightarrow \frac{1}{u} = x - \ln y$$

sol'n

$$\boxed{u = \frac{1}{x - \ln y}}$$

ex 3 $\quad y u_x - x u_y = u$

$u_s = u_x x_s + u_y y_s$

if $\quad x_s = y \quad$ & ?

$\quad y_s = -x$

$\quad u_s = u \Rightarrow u = c(r) e^s$

$x_{ss} = y_s = -x \quad$ so $\quad x_{ss} + x = 0 \qquad \left(y'' + y = 0 \right)$

$x = a(r) \sin s + b(r) \cos s$

$y = a(r) \cos s - b(r) \sin s$

so how do we get rid of s

and then r?