Math 3331 - ODE's

we are now starting to solve 1st order ODE's

$$y' = F(x,y)$$

we started with separable. These are eq^n's of the form

$$\frac{dy}{dx} = f(x)\, g(y)$$

As the name suggests - we separate so

$$\frac{dy}{g(y)} = f(x)dx$$

$$\int \frac{dy}{g(y)} = \int f(x)dx + c$$

ex $\frac{dy}{dx} = 2xy$ so $\frac{dy}{y} = 2x\,dx$

$$\int \frac{dy}{y} = \int 2x\,dx + \ln c \Rightarrow \ln|y| = x^2 + \ln c$$

$$e^{\ln|y|} = e^{x^2} \cdot e^{\ln c} \Rightarrow y = c e^{x^2}$$

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Next Linear. These ODE's are of the form

$$a(x)\,y' + b(x)y = g(x)$$

or $$y' + \frac{b(x)}{a(x)}\,y = \frac{g(x)}{a(x)}$$

or $$y' + p(x)y = q(x)$$ ← this we call standard form

Consider

$$\frac{dy}{dx} + \frac{y}{x} = 2$$

if we multiply by x

then $$x\left(\frac{dy}{dx} + \frac{y}{x}\right) = 2x$$

$$x\frac{dy}{dx} + y = 2x$$

or $$\frac{d}{dx}(xy) = 2x$$ ← this separates

$$d(xy) = 2x\,dx \qquad \int d(xy) = \int 2x\,dx + c$$

$$xy = x^2 + c \quad \text{or} \quad y = x + c/x$$ // sol^n

consider

$$\frac{dy}{dx} - \frac{y}{x} = 6$$

if we multiply by $\frac{1}{x}$

so $$\frac{1}{x}\left(\frac{dy}{dx} - \frac{y}{x}\right) = \frac{6}{x}$$

$$\underbrace{\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y} = \frac{6}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{6}{x}$$ ← this separates

consider $$\frac{dy}{dx} + 2y = 1$$ ← this can separate right now but

if we mult. by e^{2x}

$$e^{2x}\left(\underbrace{\frac{dy}{dx} + 2y}\right) = e^{2x}$$

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}$$

Let's consider the 3 examples

(1) $\frac{dy}{dx} + \frac{y}{x} = 2$ (2) $\frac{dy}{dx} - \frac{y}{x} = 6$ (3) $\frac{dy}{dx} + 2y = 1$

mult by μ - "integrating" factor

$\mu = x$ $\mu = \frac{1}{x}$ $\mu = e^{2x}$

then $\frac{d}{dx}(xy) = 2x$ $\frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{6}{x}$ $\frac{d}{dx}(e^{2x}y) = e^{2x}$

the integrating factor makes inside and we get an ODE ~~that~~ separates

so - how do we come up with μ

Consider $\frac{dy}{dx} + p(x)y = g(x)$

$$\mu \underbrace{\left(\frac{dy}{dx} + p(x)y\right)} = \mu g(x)$$

$$\frac{d}{dx}(\mu y) = \mu g(x)$$ ← separable

Now let's expand this

if $\mu\left(\frac{dy}{dx}+py\right) = \frac{d}{dx}(\mu y)$

$$= \mu\frac{dy}{dx} + \frac{d\mu}{dx}y$$

$\Rightarrow \quad \mu p \cancel{y} = \frac{d\mu}{dx}\cancel{y}$ ← sep

so $\quad \frac{d\mu}{\mu} = p(x)\,dx$

$$\int \frac{d\mu}{\mu} = p(x)dx \Rightarrow \ln|\mu| = \int p(x)dx$$

so $\quad \mu = e^{\int p(x)dx}$ ← the integrating factor

check

$$\frac{dy}{dx} + \frac{y}{x} = 2 \qquad p(x) = \frac{1}{x}$$

$$\mu = e^{\int \frac{dx}{x}} = e^{\ln|x|} = x$$

$$\frac{dy}{dx} - \frac{y}{x} = 6$$

$$p(x) = -\frac{1}{x} \qquad \mu = e^{\int -\frac{dx}{x}} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

so $\mu = \frac{1}{x}$

Ex $\frac{dy}{dx} + \frac{2y}{x} = 1$

$$P(x) = \frac{2}{x} \qquad \mu = e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2$$

$$x^2\left(\frac{dy}{dx} + \frac{2y}{x}\right) = x^2$$

$\frac{d}{dx}(x^2 y) = x^2$ ← this separates