

1. **Three-Player Chess (20 pts)** Consider a three-player game (similar to chess), where players alternate moves down a tree (player 1 (Red) moves first, then player 2 (Blue), then player 3 (Green), then Red, etc.), where every move sequence ultimately leads to a terminal state in a finite number of steps. Suppose every terminal state is one of the following: Win for Red, Win for Blue, Win for Green, Draw.

Now consider a modification of this game where both the moves of Blue and Green are taken by a new player, BlueGreen. If the original game would have been a win for Red or a draw, then the same outcome is reached in the new game. If Blue or Green would have won the original game, then BlueGreen wins the new game. Note that while the original game had three players, the new version has only two players (though it still differs from standard chess, as player BlueGreen moves twice in a row between player Red moves).

Prove that exactly one of the following is true in the new game:

- (a) Red has a winning strategy.
  - (b) BlueGreen has a winning strategy.
  - (c) Both players have strategies that guarantee at least draws.
2. **“Big Win” Chess (40 pts)** Consider a game very similar to chess, but with the following modification. If White successfully attacks Black King, and White still has his Queen remaining while Black does not have his Queen, then the outcome is a “Big Win” for White. The player who wins big will receive a slightly higher payoff from winning: a Win is worth 1 while a Big Win is worth 2 (and a tie is worth 0).

- (a) Prove that exactly one of the following is true (20 pts):
    - i. White has a strategy that guarantees exactly a Big Win.
    - ii. White has a strategy that guarantees AT LEAST a Win (i.e., Win or Big Win), but does not guarantee a Big Win.
    - iii. Black has a strategy that guarantees exactly a Big Win.
    - iv. Black has a strategy that guarantees AT LEAST a Win (i.e., Win or Big Win), but does not guarantee a Big Win.
    - v. Both players have strategies that guarantee at least ties.
  - (b) Prove that exactly one of the following is true (20 pts):
    - i. White has a strategy that guarantees AT LEAST a Win (i.e., either a Win or a Big Win)
    - ii. Black has a strategy that guarantees AT LEAST a Win (i.e., either a Win or a Big Win)
    - iii. Both players have strategies that guarantee at least ties.
3. **Not von Neuman (40 pts)** Consider a finite game that has three possible outcomes (Win for White, Win for Black, Tie).

- (a) Consider a hypothetical “theorem” that exactly one of the following is true:
  - i. White has a strategy that guarantees a Win.
  - ii. Black has a strategy that guarantees a Win.
  - iii. Both players have strategies that guarantee exactly ties.

Show that this hypothetical theorem is false (20 pts).

- (b) Consider a hypothetical “theorem” that exactly one of the following is true:
  - i. White has a strategy that guarantees at least a tie.
  - ii. Black has a strategy that guarantees at least a tie.

Show that this hypothetical theorem is false (20 pts).