Math 6345 - Adv. ODE's

Consider the system

$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3$$

the only critical pt is $(0,0)$

Linear System

$$D_x f = \begin{pmatrix} 1-3x^2 & -1 \\ 1 & 1-3y^2 \end{pmatrix}$$

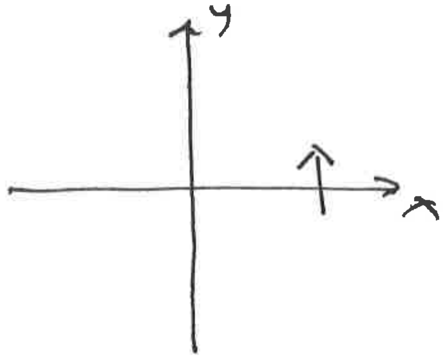
and at $(0,0)$

$$\dot{\bar{x}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \bar{x}$$

Eigenvalues $\begin{vmatrix} \lambda-1 & 1 \\ -1 & \lambda-1 \end{vmatrix} = 0 \qquad (\lambda-1)^2 + 1 = 0$

$\lambda = 1 \pm i$

so the linear system predicts a unstable spiral



at $(1,0)$

$\dot{x} = 0, \ \dot{y} = 1$

so spirals outward ccw.

Lypunov Function

$$V = x^2 + y^2$$

$$\dot V = 2x\dot x + 2y\dot y$$

$$= 2x(x - y - x^3) + 2y(x + y - y^3)$$

$$= 2x^2 - \cancel{2xy} - 2x^4 + \cancel{2xy} + 2y^2 - 2y^4$$

$$= 2(x^2 + y^2 - x^4 - y^4)$$

If

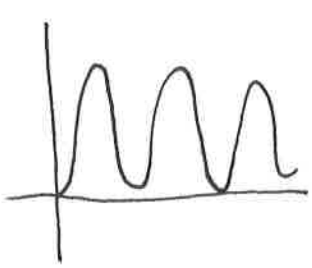
$x = r\cos\theta$, $y = r\sin\theta$

then

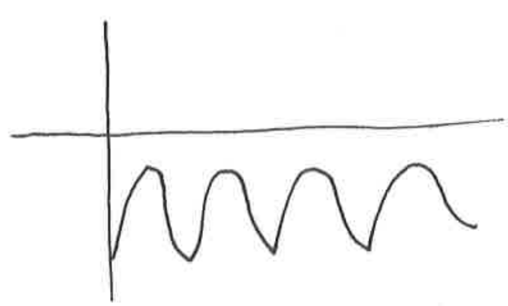
$$\dot V = 2\left(r^2 - r^4(\cos^4\theta + \sin^4\theta)\right)$$

as we vary the r the sign of $\dot V$ will change

$r=1$



$r=2$



so what value of r is $\dot{V} \leq 0$

so let's find out when $\dot{V}$ takes it max

$$\frac{d\dot{V}}{d\theta} = -r^4\left[-4\cos^3\theta\sin\theta + 4\sin^3\theta\cos\theta\right]$$

$$= 4r^4 \sin\theta\cos\theta\left(\cos^2\theta - \sin^2\theta\right)$$

$$= 2r^4 \sin 2\theta \cos 2\theta$$

$$= r^4 \sin 4\theta$$

$\frac{d\dot{V}}{d\theta} = 0$ when $4\theta = n\pi$ $\quad\theta = n\pi/4$

$\theta = 0, \pi/4, \pi/2 \; \ldots$

so $\dot{V}_{max} = 2\left[r^2 - r^4\left(\cos^4\frac{\pi}{4} + \sin^4\frac{\pi}{4}\right)\right]$

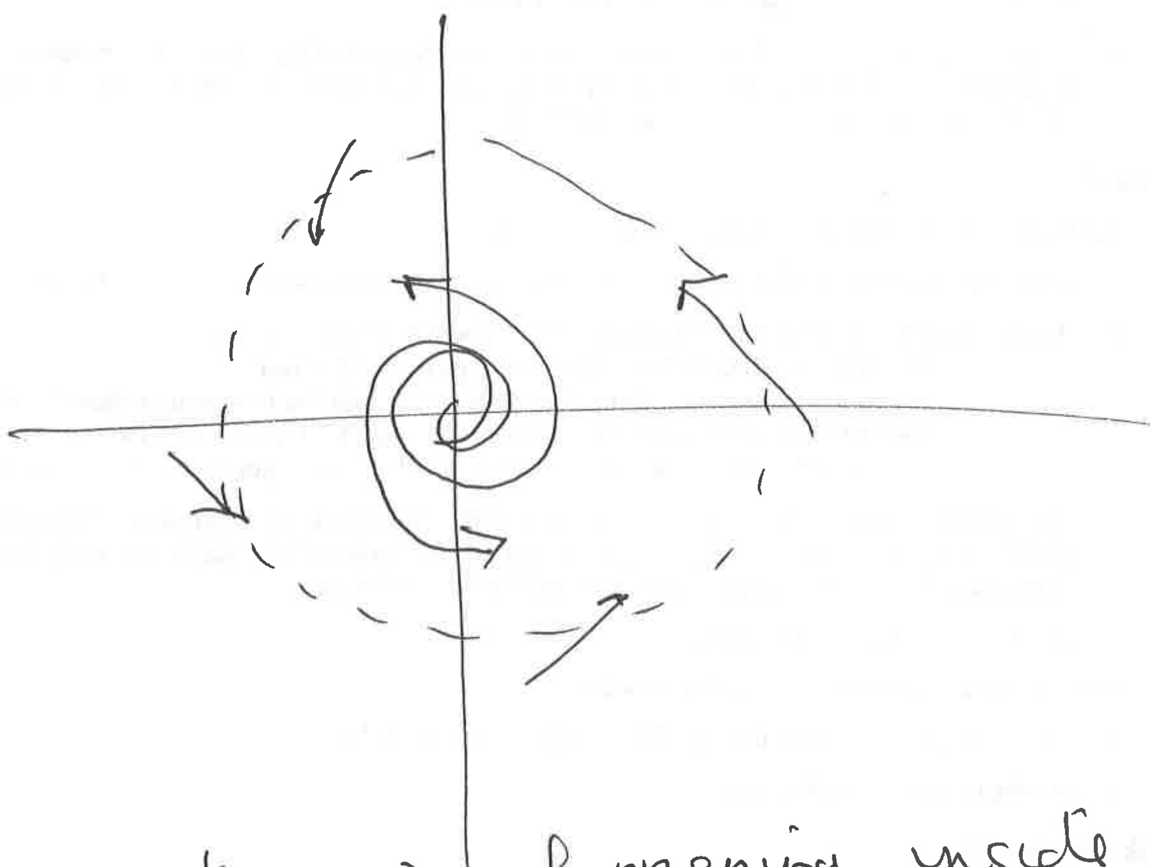
$$= 2\left[r^2 - r^4\left(\left(\frac{1}{\sqrt{2}}\right)^4 + \left(\frac{1}{\sqrt{2}}\right)^4\right)\right]$$

$\dot{V}_{max} = 2\left(r^2 - \frac{r^4}{2}\right)$ $\quad\dot{V}_{max} = 0$ when $r = \sqrt{2}$

so this is what we have

on a circle $x^2+y^2=2$ $(r=\sqrt{2})$

$\frac{dV}{dt} \le 0$ meaning the flow is in the circle and we saw that the origin was an unstable spiral - unstable



so something is happening inside this circle!

Limit Cycle

An isolated closed trajectory or path

If all neighboring trajectories approach a limit cycle we say the limit cycle is stable otherwise it's unstable. Also possible to be semi stable.

Note: Limit cycles cannot occur in linear systems

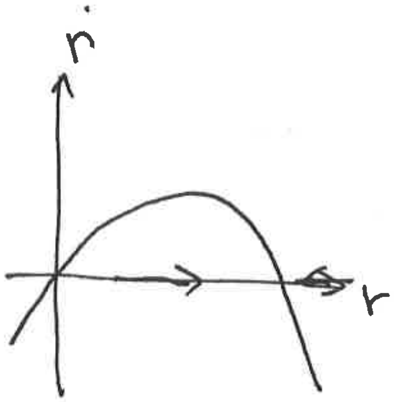
Ex
$$\dot{x} = x - y - x\sqrt{x^2+y^2}$$
$$\dot{y} = x + y - y\sqrt{x^2+y^2}$$

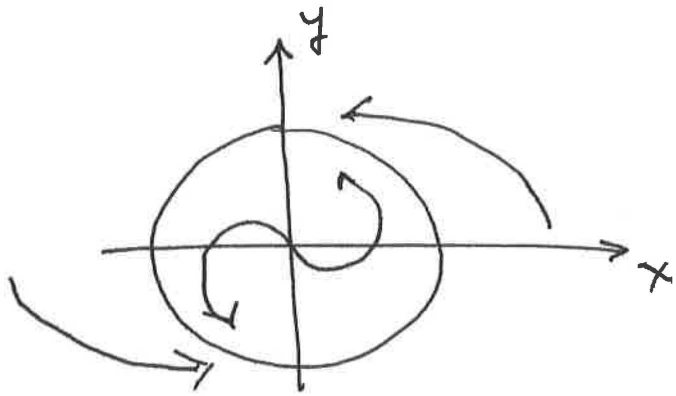
Switch to polar

$$r\dot{r} = x\dot{x} + y\dot{y}$$
$$= x\left(x - y - x\sqrt{x^2+y^2}\right) + y\left(x + y - y\sqrt{x^2+y^2}\right)$$
$$= x^2 + y^2 - x^2\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}$$
$$= x^2+y^2\left(1 - \sqrt{x^2+y^2}\right) = r^2(1-r)$$

so $\dot{r} = r(1-r)$

Furthermore $\dot{\theta} = 1$





Also, $(0,0)$ is unstable.

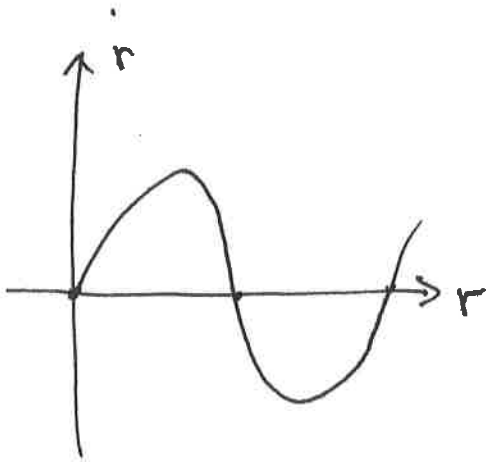
Linear System

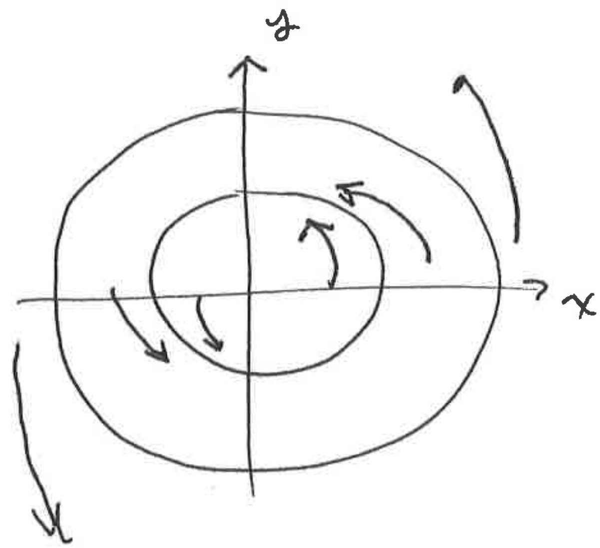
$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$|\lambda I - A| = 0$ so $\begin{vmatrix} \lambda-1 & 1 \\ -1 & \lambda-1 \end{vmatrix} = 0$ $\quad (\lambda-1)^2 + 1 = 0$

$$\lambda = 1 \pm i$$

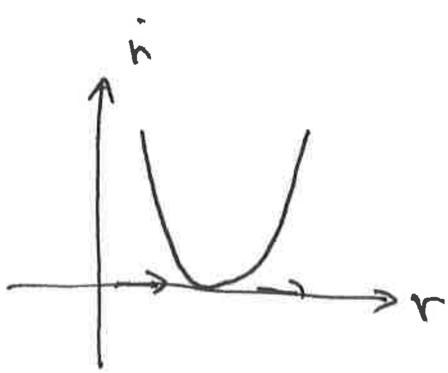
Ex $\quad \dot r = r(1-r)(2-r) \qquad \dot\theta = 1$

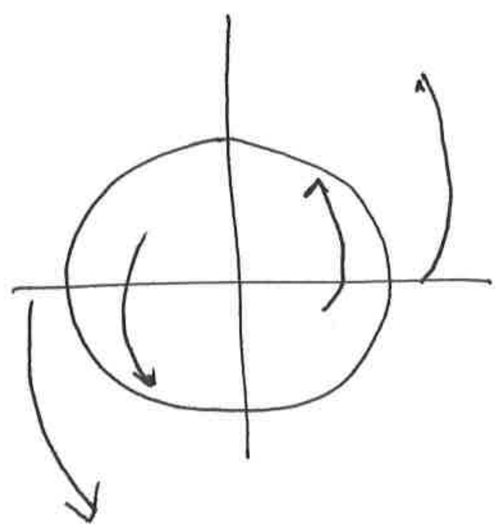




so $r=1$ is asy. stable L.C.
& $r=2$ is unstable L.C.

Ex $\quad \dot r = (1-r)^2 \qquad \dot\theta = 1$





so here is an example of a semi stable limit cycle.