Title: Sparks of Symmetry
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Mathematical Description

Intro:

Families gather around blankets laid in the green grass fields. Children run around the picnic tables full of snacks. Dusk meets night as a whistle is shot into the sky. Fireworks soar into the sky and take shape mid-air before shimmering away into the darkness. While cameras flash in an attempt to capture the image forever, that beauty lives forever in an individual's life. Gaussian periods replicate that picture-perfect moment—a beautiful, colorful, symmetry contrasted against the black night sky.

Have you ever heard of the Prince of Math? Johann Carl Friedrich Gauss (1777-1855) was a German mathematician and physicist who made tremendous contributions in numerous fields. Because of these Gauss showed his talent from a young age—in early elementary school he amazed his teacher by summing the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers, each pair summing to 101. Gauss contributed incredible knowledge to math and science—so much so that he wouldn't even teach his sons mathematics in fear of them not living up to his greatness! Gaussian Periods emerged when Gauss began theorizing compass and straightedge construction of shapes. Who would have thought that these complex images emerged from simple play with lines and angles! We can never know what inspired Gauss to pursue math and make revelations that still affect today. It could very well be inspired by seeing a firework burst and dissolve into the night sky.

Mathematical Description:

So what *is* a Gaussian period? Gaussian periods were first introduced and studied by Carl Friedrich Gauss and used in his work on the constructability of regular polygons, as well as his work in number theory. Given a positive integer n, an integer ω coprime to n, and an integer k also coprime to n, the *Gaussian period of modulus n and generator* ω, denoted $\eta_{n,\omega,k}$, is defined to be

$$\eta_{n,\omega,k} := \sum_{j=0}^{d-1} e^{2\pi i \omega^j k/n},$$

where d is the multiplicative order of ω modulo n (that is, d is the smallest positive integer ≥ 1 such that $\omega^d \equiv 1 \pmod{n}$). Each of our visualizations depicts a plot in the complex plane of the set

$$\mathsf{G}(n,\omega) := \{\eta_{n,\omega,k} \mid k = 1, 2, \ldots, n\}$$

for various inputs of n and ω. For example, the super detailed one is a plot of $\mathsf{G}(n,\omega)$ with $n = 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, and $\omega = 29 \cdot 41$. We plotted these using the app `Gaussian Periods`, created by E. Eischen with assistance from R. Lipshitz. The app is freely available for download at http://www.elleneischen.com/gaussianperiods.html, and we highly encourage the reader to download the app and make Gaussian periods of their own! Instructions for download can be found on the webpage. We do note however that the app is currently only supported on MacOS (Apple computers). For a more detailed account of how the app works, we refer the reader to [1].

The coloring scheme works as follows: fix a positive integer $c \mid n$ (called the *color modulus*), and for $j = 1, 2, \ldots, c$, assign the same color to all points in the set $\{\eta_{n,\omega,k} \mid k \equiv j \pmod{c}\}$. As it turns out, the coloring can determined by considering the orbits induced by the action of $\langle \omega \rangle$ on the set $\mathbb{Z}/n\mathbb{Z}$. Particularly, the orbits of this action partition $\mathbb{Z}/n\mathbb{Z}$ into disjoint subsets, which correspond to the points appearing in $\mathsf{G}(n, \omega)$. We are then able to determine which points are assigned the same colors by looking at points in $\mathsf{G}(n, \omega)$ and considering their values of k. Points with values of k that are equivalent modulo c are then assigned the same color.

Outside of producing neat looking visualizations when computed at large scales, Gaussian periods also have applications in various other areas of math. Gaussian periods are closely related to Gauss sums, which are certain sums of roots of unity that appear all over the place, particularly in analytic number theory. The particular sums Gauss was working with are now known as quadratic Gauss sums. These sums proved to be quite powerful and have since been generalized quite a bit. In particular, (quadratic) Gauss sums can be used to prove the law of quadratic reciprocity, one of the landmark theorems of elementary number theory. If the reader would like to see such a proof, we refer them to https://wstein.org/edu/2007/spring/ent/ent-html/node54.html. With a bit more work, Gauss sums can also be used to prove cubic and quartic reciprocity. Gauss also used his sums in his work on the constructibility of regular polygons. With this work, Gauss was finally able to put to rest some classic geometry problems dating back to the ancient Greeks whose solutions had evaded mathematicians for hundreds of years. Two of the most famous of these problems are: "Are you able to construct a square with the same area as a circle using only a straightedge and compass?" and "Are you able to trisect (i.e. split into three equal parts) an angle using only straightedge and compass?". We now know that the answer is "no" to both of these, thanks to Gauss. Gauss also managed to prove that you are able to construct a regular n-gon using a straightedge and compass if and only if n is the product of a power of two and any number of primes of the form $2^{2^m} + 1$ (such primes are called *Fermat primes*, named after French mathematician Pierre de Fermat).

To learn more about these topics, we recommend the elementary number theory sequence, Math 347-348, and for more advanced applications, the abstract algebra sequence Math 444-446, particularly Math 446 (though you need 444 and 445 as prerequisites).

References:

[1] E. Eischen and S. R. Garcia. *A Gallery of Gaussian Periods*. 2020. https://archive.bridgesmathart.org/2020/bridges2020-243.pdf.

[2] P. Roberge. *Carl Friedrich Gauss*. Corrosion Doctors, `https://corrosion-doctors.org/Biographies/GaussBio.htm`.