The Fractional Calculus, Fractional Differential Equations, and Laplace Transform

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Abstract - This research review study paper explores the possibility of applying the Laplace transform for solving linear fractional differential equations from several sources, academic articles and journals. The Laplace transform is a very powerful component in engineering, science, and applied mathematics. It permits to transform the fractional differential equation into the algebraic equation, so as to solve the algebraic equations to obtain the unknown value as its function, and that further can be processed by applying the Inverse Laplace Transform.

The subject applications of fractional calculus, which means, calculus of integrals as well as derivatives of some arbitrary real and complex order, have possessed seemingly high reputation in the past 30 years, specifically because of their established applications in innumerable diverse fields of engineering and science. Certain areas of contemporary fractional model applications involve Fluid Flow, Dynamical Processes, Diffusive Transport close to Diffusion, Solute Transport in Similar to Porous Structures, Electromagnetic Theory, Viscoelastic Material Theory, Earthquake Dynamics, Dynamical Control Theory Systems, Bioscience, Signal and Optical Processing, Geology, Economics, Astrophysics, Chemical Physics, Statistics, Probability and so on.

I. INTRODUCTION

Even though fractional derivatives carry a lengthy history in mathematics, their multiple definitions of nonequivalent fractional derivatives are responsible of their non usage (Podlubny, 1999). Another problem is that the fractional derivatives do not carry clear geometrical interpretation due to the nonlocal characteristics (Podlubny, 2002). But, the physics and mathematics have proved its need, especially due to its interdisciplinary application that can be conveniently formatted by using fractional derivatives. To give an example, the earthquake, nonlinear oscillation can be assessed only with fractional derivatives (He, 1998). Again, it has been used the traffic model of fluid-dynamics using fractional derivatives ((He, 1999). This has completely eliminated the shortage arose out of various assumptions made so far in the continuous traffic flow circumstances. Based on experimental data available, the equations of fractional partial differential for the porous media seepage flow are also suggested, and the fractional order differential equations proved to be the most vital tool model several other physical phenomena. Further the fractional derivative review and applications in statistical and continuum were produced, while the analytical results on the uniqueness and subsistence of relevant fractional differential solutions to the equations were investigated by several authors (Grigorenko & Grigorenko, 2003). Many fractional differential methods were used to solve equations, along with fractional differential and partial equations, fractional integral and differential equations, moreover, the dynamic systems involving fractional derivatives, like Adomian's method of decomposition; Variation method of iteration; Homotopy method of perturbation; Homotopy method of analysis; and certain spectral methods (Momani & Noor, 2006).

II. DIFFERENTIAL EQUATIONS

In the branch of fractional calculus, mathematical analysis finds several ways to define the powers of real or complex numbers of *D* – the differentiation operator.

D f (x) = d / dx f (x), and of the integration operator *J* (J symbol, 2014).

J f (x) = ∫ 0 x, f(s) ds, and

Calculus development of such equations is generalized as classical one.

The *powers* term indicates linear operator iterative application of a function, in different analogy called functional composition acts like a variable,

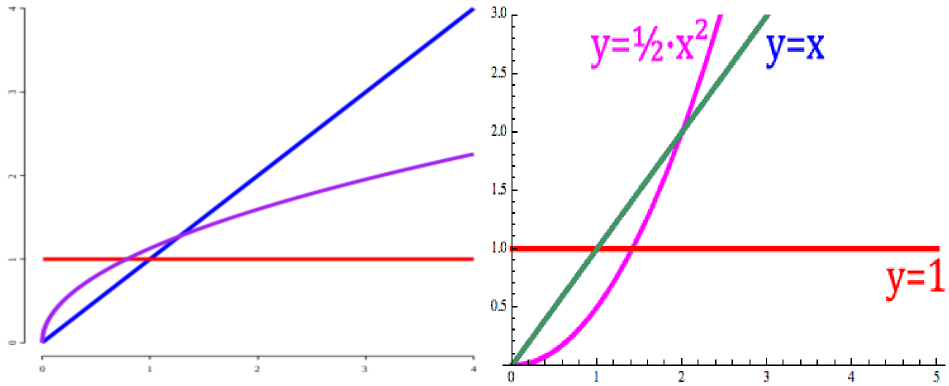
i.e. $f^{\circ 2}(x) = f \circ f(x) = f(f(x))$.

To give an example, it can be meaningfully interpreted as

D = D 1 2

Acting as the square root of a function analogue for the differentiation operator, which develops the elements in the similar space, that means, a certain linear operator expression applied *twice* for any different function and that produces the similar effect like differentiation. Otherwise, the D - linear functional and be specified for each *a* – the real-number in a way, when *a* acts like a value $n \in \mathbb{Z}$ of the integers that coincides normal *n*-fold *D* differentiation when $n > 0$, and when $n < 0$, the $-n$ th power of *J*. The basic motivation for introducing these types of differentiation *D* operator is because of the operator power sets $\{ D^a \mid a \in \mathbb{R}\}$ which is defined as *continuous* semi-group having *a* as the parameter, the original semi group *discrete* of $\{D^n \mid n \in \mathbb{Z}\}$ for *n* integer, which is a denumerable sub-group. The constant formation of semi-groups works like well generated mathematical theory, which can be applied to other mathematical branches of various Different Fractional types of equations, called extraordinary differential equation, which are generalized differential equations formed by using fractional calculus (Kilbas, et al., 2006).

III. HALF DERIVATIVE SIMPLE FUNCTION



The above Maroon curve- half derivative, Blue curve- function y = x, Red line- first derivative

Another figure Green line derivative operator, oscillates between the derivative ($\alpha = +1$: $y = 1$) and anti-derivative ($\alpha = -1$: $y = \frac{1}{2} \cdot x^2$), while $y = x$ is the simple continuous power function (Kilbas, et al., 2006).

IV. FRACTIONAL ORDER DIFFERENTIAL (FOD) EQUATIONS

The mathematical part of fractional differential equation and solution methods elaborated by various authors using series methods, iteration methods, Fourier transforms techniques, fractional differential equation, special methods to obtain rational equations of special kind, the techniques of Laplace transform, operational calculus process. Otherwise, the mathematical features of fractional differential methods and equation and their solutions were discussed by many authors, indicating the iteration method, the series method, the Fourier transform technique, special methods for fractional differential equations of rational order or for equations of special type, the Laplace transform technique and the operational calculus method. Many mathematical methods involving variation iteration methods, Adomian decomposition methods, and Homotopy perturbation methods also were employed to get the correct analytic solution (Ghosh, Sarkar & Das, 2016).

V. FRACTIONAL CALCULUS

Fractional calculus is a mathematical analysis branch of studies that uses the possible applications and power of real number or any complex numbers for the differential operator $D = d / dx$; Along with *J* integration operator as $\sqrt{D} = D^{1/2}$, as a differentiation operator square root, called as operator of half iterate, which s the expression for certain operator when functional twice to that function will carry the similar effect as differentiation, while the powers are referred as the iterative composition or application in the similar sense as f 2 x = f (f (x))

VI. CAPUTO FRACTIONAL DERIVATIVE

To generalize, several methods of fractional order differentiations are applied, like Gr Unwald-Letnikov, Riemann-Liouville, Caputo, and other generalized functions. Mostly, the derivative of Riemann-Liouville fraction is used by academicians, however, their problem solving approach is not favorable for the virtual physical problem solving process. This because, it needs a definite explanation with clarity of initial conditions of fractional order, which has no explanation with virtual meaning. Therefore, Caputo had introduced another definition, which carries defining integer advantage of order of initial conditions of the differential equation fractional order.

- $D_*^\alpha f(t) := J^{m-\alpha} D^m f(t)$ with $m-1 < \alpha \leq m$, namely

$$D_*^\alpha f(t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases}$$

In such context, the *powers* are the iterative application of linear operator that acts on this function, acting on variables like $f^2(x) = f(f(x))$. In fact, the processes in the real-world are normally of the system of fractional order, and the derivatives are in the Caputo sense. The parameters describing the fractional derivatives can vary to acquire several responses. Therefore, Caputo established another definition, having an advantage to define the integer for the initial conditions of the differential equation fractional order, and mainly applied in numerical algorithms, basically defined as Caputo fractional-order integration with differentiation (Ghosh et al., 2016).

VII. THE CAPUTO FRACTIONAL-ORDER INTEGRATION AND DIFFERENTIATION

Fractional calculus fundamentals and Special functions

The mathematical field of fractional calculus develops from conventional calculus integral definitions and derivative operators similar to fractional exponents of outgrowth exponents having an integer value (Da, 2008).

Fractional calculus uses engineering and physical models to process the best way to describe using fractional differential equations. There are efficient and reliable techniques to obtain the fractional differential equations applying the numerical solutions, and the fractional derivatives are elaborated in the Caputo sense. The main features of this approach is that, it reduces all the problems of solving algebraic equation system and that highly simplifies the problem, by using methods like linear and also nonlinear fractional differential equation to solve the problems (Saadatmandia & Dehgha, 2010).

The fractional calculus concept, also called fractional derivatives, fractional integral, generalizes and unifies n-fold integration and integer-order differentiation notions and the Caputo fractional order derivative was defined as

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt, \quad n-1<\alpha\le n, n\in\mathbb{N},$$

where $\alpha > 0$ is the order of the derivative and n is the smallest integer greater than α. For the Ca

$$D^{\alpha}C = 0, \quad (C \text{ is a constant}),$$

$$D^{\alpha}x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil\alpha\rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \ge \lceil\alpha\rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor\alpha\rfloor. \end{cases}$$

Figure: Fractional-order derivative, Caputo definition (Saadatmandia & Dehgha, 2010).

VIII. LAPLACE TRANSFORM

The Laplace transform is a mathematical technique, well recognized and useful to solve differential equations. It is designated as a respect for Pierre Simon De Laplace, the French mathematician, (1749-1827). Like any other, the Laplace transform alters one to another signal as specified in the set of equations and rules.

Several mathematical equations and problems use the transformation method, basically to convert one problem in one different problem easy to solve. The moment a solution is achieved, the method of the inverse transform is applied to get the solution to the basic and original problem. The Laplace transform acts like a vital tool to make results of linear constant coefficient with differential equations very simple.

The Laplace transform helps convert to algebraic equations from differential equations, that are easy to maneuver and solve. As soon as a proper solution is found in its domain, the inverse transform is applied to get the results of differential equations. Laplace transforms have become the most crucial tool to study the linear time inversion systems.

The Laplace transform comes into use when time domain is complex, and seldom requires signal processing. For every practical application, the signal of time domain is absolutely real.

Definition

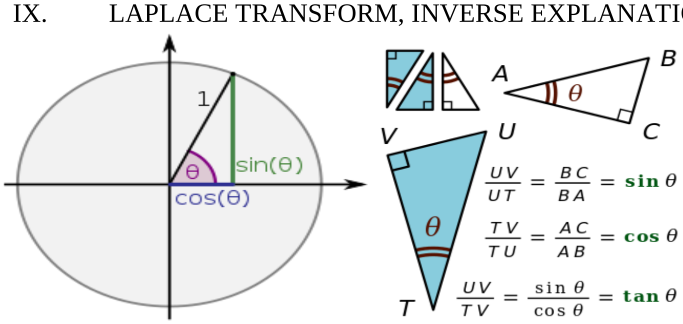
If f (t) be shown as t ≥ 0 and assuming the function meets specific conditions, the Laplace transform f (t), can be denoted as F (t) or F (s), and further identified using the equation

F (*s*) =∫ ∞ *of* (*t*) *e*− *st dt*

When the inappropriate integral congregates. K is symbolized as the Laplace transform kernel and is shown as (*s*, *t*) =*e* – *st* <

If we compute the Laplace transform as the prominent elementary function, in the absence of any restrictions imposed on f (t), hence, it has a Laplace transform.

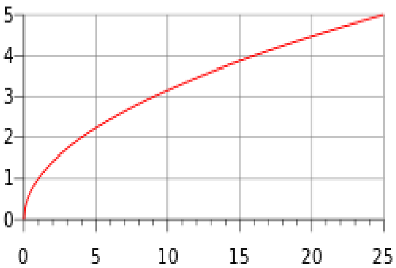
IX. LAPLACE TRANSFORM, INVERSE EXPLANATION

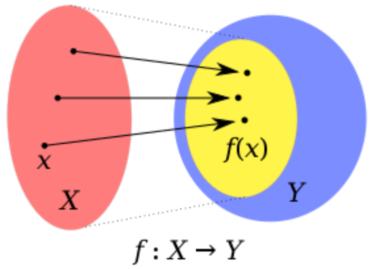


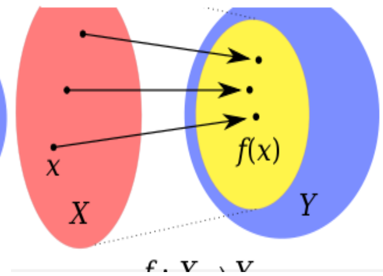
In the fundamentals of trigonometry, when two right angles are of equal acute angles, they are identical, hence, all sides are same. Hence, the Portional constant can be written as: sin θ, tan θ, cos θ, when, θ becomes a standard measure, five of those acute angles.

In mathematical Trignometry knowledge study, like structure, space, quantity and change, the use of inverse Laplace transform acts like a $F(s)$ function, an exponentially-restricted, and piecewise-consistent real function f(t) indicating its property.

L { f } (s) = L { f (t) } (s) = F (s), Where, L is the Laplace transform, an integral transform,







The square root, real-value function, $f(x) = \sqrt{x}$, where the domain is consisting of real nonnegative numbers

F is a pink domain function *X*, transferred towards blue domain *Y*. Inside *Y* yellow oval is the *f* image. Both images are known as *f* range, and the function gives an output value for every domain member. Otherwise, the value functions taken as output as the function image, sometimes referred as function range (Paley, & Weichsel, 1966).

When *F* (*s*) function carries an inverse *f* (*t*) Laplace transform, in that case, *f* (*t*) is distinctly considered as the function, which differs from other at the set point having a zero measure.

The Laplace transform together with inverse Laplace transform hold beneficial properties, useful for to analyze the linear dynamic systems (Rosenbaum, et al., 1984).

X. DISCUSSION & CONCLUSION

A latest generalization functional methods have been generated for the linear differential equation having fractional derivative. This newest generalization is derived from the Caputo fractional derivatives. Hence, it can be stated that the technique applied is highly effective, strong, and efficient for attaining the analytical solution for a big category of equations, linear differential kind of fractional order. Hence, this research review offers a Laplace Transform overview. The basic application of Laplace Transform is to convert the time functional domain into the frequency domain. The Laplace Transform main properties and its other special functions mentioned include the Laplace Transform, Inverse explanation, which is a very useful and effective mathematical component that simplifies many complex problems in the control and stability zone. In effect, the Laplace Transform is provided incredible applications and help in several power engineering, chemical, electrical, physical and applied sciences.

XI. REFERENCES

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