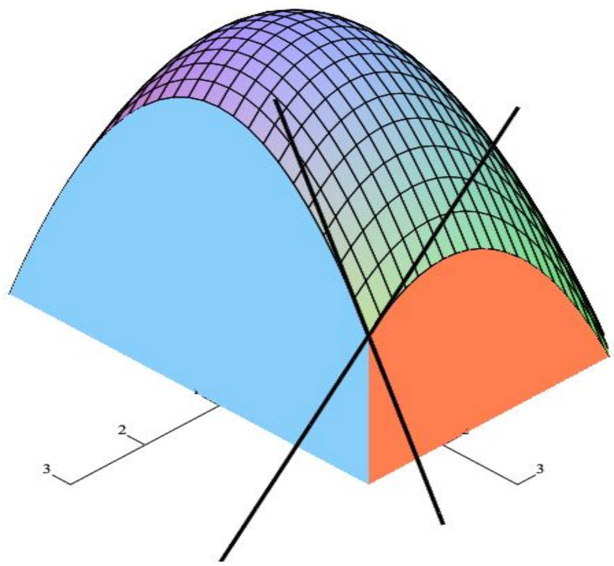
Calculus 3 - Surface Integrals over Vector Fields

Recall the section on the tangent plane. We created two vectors with two tangent lines. If the surface is $z = f(x, y)$ then the tangent vectors are

$$\vec{u} =< 1, 0, f_x >, \quad \vec{v} =< 0, 1, f_y > . \tag{1}$$

We evaluate these at some point (a, b).



We now cross these two vectors to get the normal so

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} =< -f_x(a,b), -f_y(a,b), 1 > . \tag{2}$$

The equation of the tangent plane is then (where $c = f(a, b)$)

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z-c) = 0 \tag{3}$$

or

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z-c) = 0. \tag{4}$$

If the surface is given implicitly, say by $G(x, y, z) = 0$, then the partial derivatives are given as

$$z_x = -\frac{G_x}{G_z}, \quad z_y = -\frac{G_y}{G_z} \tag{5}$$

and the equation of the tangent plane is

$$-z_x(z, b)(x-a) - z_y(a, b)(y-b) + (z-c) = 0 \tag{6}$$

or

$$G_x(a, b, c)(x-a) + G_y(a, b, c)(y-b) + G_z(a, b, c)(z-c) = 0 \tag{7}$$

Unit Normal to Surface

We now define the unit normal $\vec{N}$ to a surface given by $G(x, y, z) = 0$ as

$$\vec{N} = \frac{\nabla G}{\|\nabla G\|} \tag{8}$$

where the gradient of G is given by $\nabla G = \langle G_x, G_y, G_z \rangle$. We can also orient the normal as to point outward or inward and would simply multiply (8) by -1.

Example 1. Find the unit normal to the unit sphere

$$x^2 + y^2 + z^2 = 1 \tag{9}$$

Here $G = x^2 + y^2 + z^2 - 1$ so $\nabla G = \langle 2x, 2y, 2z \rangle$

$$\|\nabla G\| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4(x^2 + y^2 + z^2)} = 2 \tag{10}$$

and the unit normal is $\overrightarrow{N} = \dfrac{\langle 2x, 2y, 2z \rangle}{2} = \langle x, y, z \rangle$.

Flux

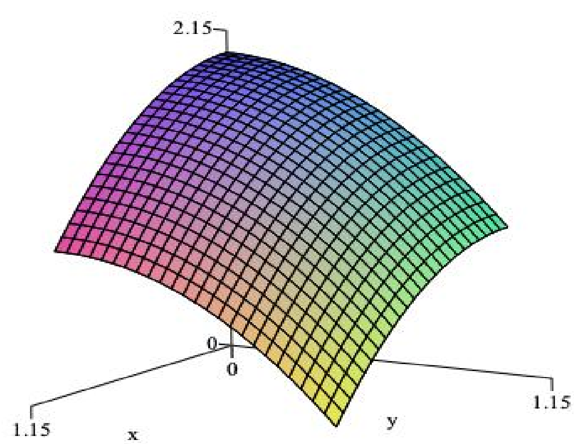
One of the principal applications of surface integrals over vector fields is fluid flow through a surface. Consider an oriented surface S submerged in a fluid having a continuous velocity field $\overrightarrow{F}$. Let dS be a small area on the surface over which $\overrightarrow{F}$ is nearly constant. Then the amount of field crossing this surface per unit time is approximated by $\overrightarrow{F} \cdot \overrightarrow{N} dS$ and adding up the element gives

$$\iint_S \overrightarrow{F} \cdot \overrightarrow{N} dS. \tag{11}$$

Def ^{n} If $\overrightarrow{F}$ is a continuous vector field defined on an oriented surface S with unit normal $\overrightarrow{N}$, then the surface integral of $\overrightarrow{F}$ over the surface S is given by

$$\iint_S \overrightarrow{F} \cdot \overrightarrow{N} dS \tag{12}$$

which is called the *Flux of* $\overrightarrow{F}$ *across* S.



Example 2. Taken from a Berkeley midterm.

Find the flux $\vec{F}$ across S where $\vec{F} = \langle x, y, z \rangle$ and the surface S is the boundary of the solid enclosed by the plane $x + y + z = 1$ and the xy, xz and yz planes.

Soln: First we find the unit normal. Since the surface is given as $x + y + z = 1$ we create G as $G = x + y + z - 1$. So $\nabla G = \langle 1, 1, 1 \rangle$ and the unit normal is given by

$$\vec{N} = \frac{\nabla G}{\|\nabla G\|} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}}. \tag{13}$$

Next, we calculate dS. Since the surface is given by $z = 1 - x - y$ then

$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dA_{xy} = \sqrt{1 + 1 + 1}\, dA_{xy}. \tag{14}$$

Now the flux integral becomes

$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{N} dS &= \iint_{R_{xy}} \langle x, y, z \rangle \cdot \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}} \sqrt{3}\, dA_{xy} \\
&= \iint_{R_{xy}} (x + y + z)\, dA_{xy} \\
&= \iint_{R_{xy}} 1 dA_{xy} \\
&= \int_0^1 \int_0^{1-x} 1 dy dx \\
&= \int_0^1 y \Big|_0^{1-x} dx \\
&= \int_0^1 (1 - x) dx \\
&= x - \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}
\end{aligned} \tag{15}$$

Example 3.

Find the flux $\vec{F}$ across S where $\vec{F} = \langle y, x, z \rangle$ and the surface S is the boundary of the solid enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Soln: First we find the unit normal. Since the surface is given as $z = 1 - x^2 - y^2$ we created G as $G = x^2 + y^2 + z - 1$. So $\nabla G = \langle 2x, 2y, 1 \rangle$ and the unit normal is given by

$$\vec{N} = \frac{\nabla G}{\|\nabla G\|} = \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{1 + 4x^2 + 4y^2}}. \tag{16}$$

Next, we calculate dS. Since the surface is given by $z = 1 - x^2 - y^2$ then

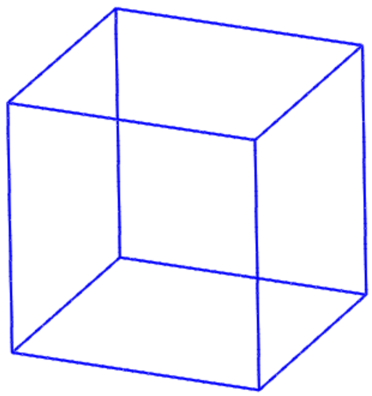
$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dA_{xy} = \sqrt{1 + 4x^2 + 4y^2}\, dA_{xy}. \tag{17}$$

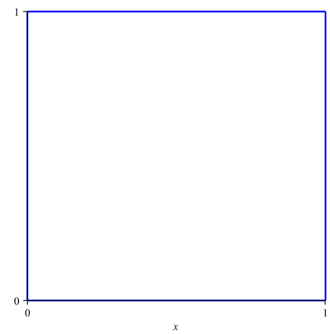
Now the flux integral becomes

$$\begin{aligned}
\iint\limits_S \vec{F} \cdot \vec{N} dS &= \iint\limits_{R_{xy}} \langle y, x, z \rangle \cdot \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + 4x^2 + 4y^2}\, dA_{xy} \\
&= \iint\limits_{R_{xy}} (4xy + z)\ dA_{xy} \\
&= \iint\limits_{R_{xy}} \left(4xy + 1 - x^2 - y^2\right)\ dA_{xy} \\
&= \int_0^{2\pi} \int_0^1 \left(4r^2 \sin\theta \cos\theta + 1 - r^2\right) r dr d\theta \\
&= \frac{\pi}{2}
\end{aligned} \tag{18}$$

Example 4. Taken from John Hopkins University.

Find the flux $\vec{F}$ across S where $\vec{F} = \langle x, xy, xyz \rangle$ through the unit cube $0 \le x \le 1, 0 \le y \le 1$ and $0 \le z \le 1$.





Soln: Since there are 6 sides to the cube we must do all 6 fluxes separately. The nice thing is that the unit normal's are easy to pick off and so are dS.

Top: Here $\vec{N} = \langle 0, 0, 1 \rangle$. Since $z = 1$, then $\vec{F} = \langle x, xy, xy \rangle$ and $\vec{F} \cdot \vec{N} = \langle 0, 0, 1 \rangle \cdot \langle x, xy, xy \rangle = xy$

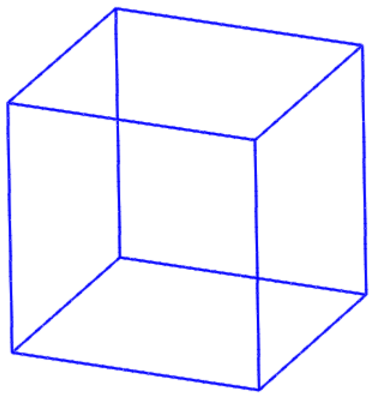
$$\iint_S \vec{F} \cdot \vec{N} dS = \int_0^1 \int_0^1 xy dy dx = \frac{1}{4} \tag{19}$$

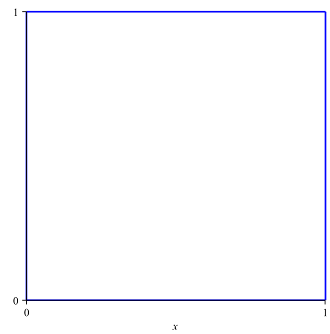
Bottom: Here $\vec{N} = \langle 0, 0, -1 \rangle$. Since $z = 0$, then $\vec{F} = \langle x, xy, 0 \rangle$ and $\vec{F} \cdot \vec{N} = \langle 0, 0, -1 \rangle \cdot \langle x, xy, 0 \rangle = 0$

$$\iint_S \vec{F} \cdot \vec{N} dS = \int_0^1 \int_0^1 0 dy dx = 0 \tag{20}$$

Right: Here $\overrightarrow{N} = \langle 0,1,0 \rangle$. Since $y = 1$, then $\overrightarrow{F} = \left\langle x, x, xz \right\rangle$ and $\overrightarrow{F} \cdot \overrightarrow{N} = \langle 0,1,0 \rangle \cdot \left\langle x, x, xz \right\rangle = xy$

$$\iint\limits_{S} \overrightarrow{F} \cdot \overrightarrow{N} dS = \int_0^1 \int_0^1 x dz dx = \frac{1}{2} \tag{21}$$



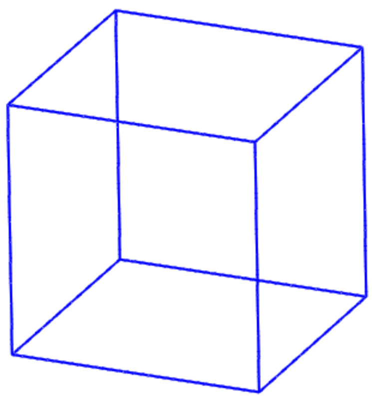


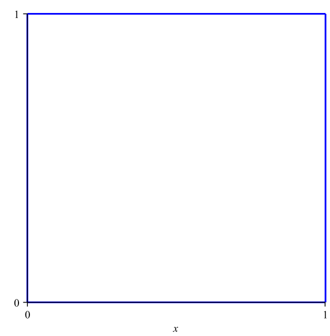
Left: Here $\overrightarrow{N} = \langle 0,-1,0 \rangle$. Since $y = 0$, then $\overrightarrow{F} = \left\langle x, 0, 0 \right\rangle$ and $\overrightarrow{F} \cdot \overrightarrow{N} = \langle 0,-1,0 \rangle \cdot \left\langle x, 0, 0 \right\rangle = 0$

$$\iint\limits_{S} \overrightarrow{F} \cdot \overrightarrow{N} dS = \int_0^1 \int_0^1 0 dz dx = 0 \tag{22}$$

Front: Here $\vec{N} = \langle 1, 0, 0 \rangle$. Since $x = 1$, then $\vec{F} = \langle 1, y, yz \rangle$ and $\vec{F} \cdot \vec{N} = \langle 1, 0, 0 \rangle \cdot \langle 1, y, yz \rangle = 1$

$$\iint_S \vec{F} \cdot \vec{N} dS = \int_0^1 \int_0^1 1 dz dy = 1 \tag{23}$$





Back: Here $\vec{N} = \langle -1, 0, 0 \rangle$. Since $x = 0$, then $\vec{F} = \langle 0, 0, 0 \rangle$ and $\vec{F} \cdot \vec{N} = \langle -1, 0, 0 \rangle \cdot \langle 0, 0, 0 \rangle = 0$

$$\iint_S \vec{F} \cdot \vec{N} dS = \int_0^1 \int_0^1 0 dz dy = 0 \tag{24}$$

So the total flux through the cube is

$$\iint_S \vec{F} \cdot \vec{N} dS = \frac{1}{4} + 0 + \frac{1}{2} + 0 + 1 + 0 = \frac{7}{4} \tag{25}$$