Math 1497 - Calc 2

Infinite Series

$$\sum_{n=1}^{\infty} a_n$$

if this converges the sum $\to$ # (a limit)

Special Series

(1) Geometric $a + ar + ar^2 + \cdots$ converges if $|r|<1$

if so $S_\infty = \dfrac{a}{1-r}$

(2) Telescopic

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - b_{n+1}$$

(1) Stop at $n = N$, (2) write out terms & cancel

(3) $N \to \infty$

(3) Harmonic $\sum_{n=1}^{\infty} \frac{1}{n}$ div

(4) p series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p>1$ conv

$p<1$ div.

Test :

(1) n^{th} term test

if $\lim_{n\to\infty} a_n = \#$ not zero the series diverges

Ex1 $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+n-1}{n^2+3n+2}$

$$\lim_{n\to\infty} \frac{3n^2+n-1}{n^2+3n+2} = \lim_{n\to\infty} \frac{3+\frac{1}{n}-\frac{1}{n^2}}{1+\frac{3}{n}+\frac{2}{n}} = 3 \neq 0$$

So the series div by nth term test.

(2) $\int$ test if $a_n = f(n)$

if (1) $f > 0$ (2) f cont^s (3) $f' < 0$

test applies

if $\displaystyle\int_1^{\infty} f(x)\,dx$ conv (div) $\displaystyle\sum_{n=1}^{\infty} a_n$ conv (div)

Ex 2 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) = \dfrac{1}{x^2+1}$ (1) $f > 0$ ✓, (2) f cont^s ✓

(3) $f' = \dfrac{-2x}{(x^2+1)^2} < 0$ ✓ so test applies

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1}x \Big|_1^b$$

$$= \lim_{b\to\infty} \tan^{-1}b - \tan^{-1}1 = \frac{\pi}{2} - \frac{\pi}{4} = \pi/4$$

so by $\int$ test, the series converges.

As we mentioned last class,

$\dfrac{1}{n^2+1}$ look like $\dfrac{1}{n^2}$

when n is large

n	$\frac{1}{n^2+1}$	$\frac{1}{n^2}$
100	.00009999	.0001
1000	9.999×10^{-7}	10×10^{-7}
10,000	9.999×10^{-9}	10×10^{-9}

this suggest we can compare the series

Test 3 Limit Comparison Test (LCT)

Let $\sum a_n$ $\sum b_n$ be series

and $\lim_{n \to \infty} \frac{a_n}{b_n} = \#$ (not zero)

if $\sum a_n$ converges $\sum b_n$ converges
if (diverges) (diverges)

Previous Ex

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \quad \text{compare w/} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \begin{matrix} p=2 \\ \text{conv} \end{matrix}$$

$$\lim_{n \to \infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2+1}} = \lim_{n \to \infty} \frac{n^2+1}{n^2} = 1$$

$\therefore$ $\sum \frac{1}{n^2}$ conv by LCT our series conv

Ex $\sum_{n=1}^{\infty} \sin \frac{1}{n}$

compare w/ $\sum \frac{1}{n}$ diverges Harmon

$$\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \text{“}\frac{0}{0}\text{”}$$

$$\lim_{x\to\infty} \frac{\cos \frac{1}{x} \left(-\cancel{\frac{1}{x^2}}\right)}{-\cancel{\frac{1}{x^2}}} = \lim_{x\to\infty} \cos \frac{1}{x} = \cos 0 = 1$$

$\therefore \sum \frac{1}{n}$ div by LCT our series div

<u>Ex</u>

$$\sum_{n=1}^{\infty} \frac{n^4 + 3n^2 + 1}{2n^5 + 6n - 2}$$

Compare w/ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{n^4 + 3n^2 + 1}{2n^5 + 6n - 2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^4 + 3n^2 + 1}{2n^5 + 6n - 2} \cdot \frac{n}{1}$$

$$\lim_{n\to\infty} \frac{n^5 + 3n^3 + n}{2n^5 + 6n - 2} = \frac{1}{2}$$

$\therefore \sum \frac{1}{n}$. div then by LCT our series div

Test #4 Direct Comparison Test (DCT)

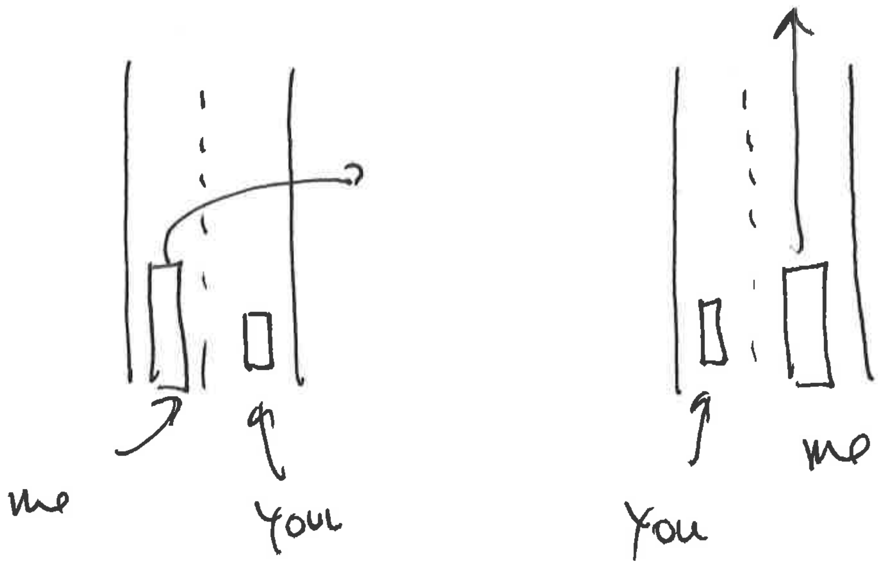
consider $\sum a_n$ $\sum b_n$

when $0 < a_n \le b_n$

if $\sum a_n$ diverges $\sum b_n$ diverges

if $\sum b_n$ converges $\sum a_n$ converges

Do car example



This series is probably the hardest since we need to have an idea of what the series will do and need an inequality.

Ex $\sum_{n\geq 1}^{\infty} \frac{1}{2^n+n}$

↓ pick this

If it converge $0 < \frac{1}{2^n+n} < ?$

it diverges $0 < ? < \frac{1}{2^n+n}$

↑ pick this

think conv

$$0 < \frac{1}{2^n+n} < \frac{1}{2^n}$$

$2^n \overset{?}{<} 2^n+n$

$0 \overset{?}{<} n$ ✓

$\therefore \sum \frac{1}{2^n}$ conv geom. with $r=1/2$

then by DCT our series conv.

Ex $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$ think div

$$? < \frac{1}{\sqrt{n-1}}$$

↑ pick that div

$\frac{1}{\sqrt{n}} \overset{?}{<} \frac{1}{\sqrt{n-1}}$

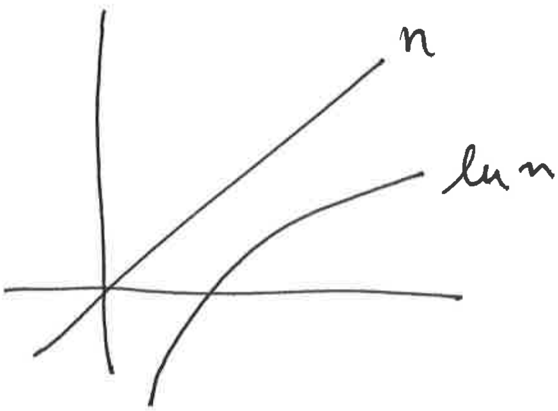
$\sqrt{n-1} \overset{?}{<} \sqrt{n}$

$-1 \overset{?}{<} 0$ ✓

$\because$ $\sum \frac{1}{\sqrt{n}}$ div $(p = 1/2)$

by the DCT our series diverges

ex $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ think div



$\ln n < n$

$\frac{1}{\ln n} > \frac{1}{n}$

or $\frac{1}{n} < \frac{1}{\ln n}$

$\because$ $\sum \frac{1}{n}$ div, then by DCT our series div