Math 3331 - ODE's

1st order ODE - Applications

1. Population Growth

A simple model is:

$$\frac{dP}{dt} \propto P$$ populations grow proportional to their size.

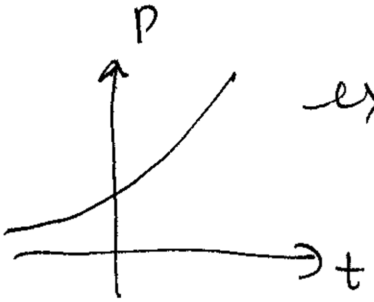
so $\frac{dP}{dt} = kP, \; P(0) = P_0$ P_0 - initial population

Sol^n: $\frac{dP}{P} = k\,dt \Rightarrow \ln|P| = kt + \ln c$

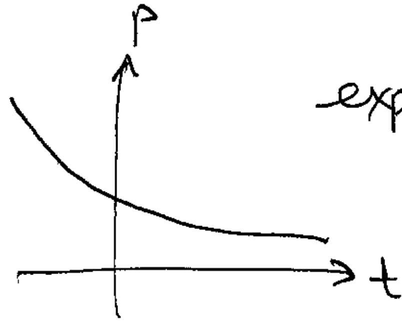
$$P = c\,e^{kt}$$

$P(0) = P_0 \Rightarrow c = P_0$ so

$$P = P_0\,e^{kt}$$



exponential growth
$k > 0$



exponential decay
$k < 0$

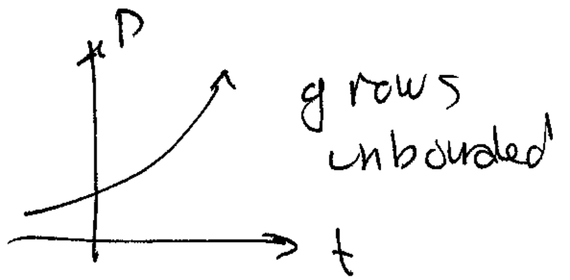
A better model - Logistic growth

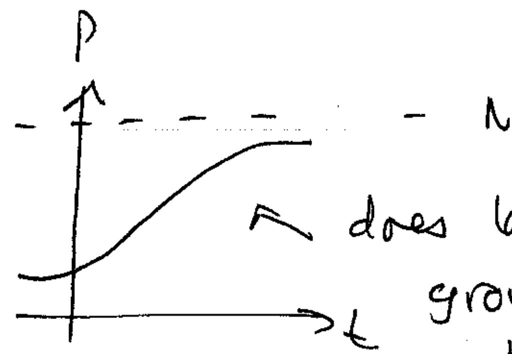
If the population ~~has~~ a limit ~~ze~~ size, say N

$$\frac{dP}{dt} \propto P, \quad \frac{dP}{dt} \propto N-P \longleftarrow \text{available spots}$$

$$\frac{dP}{dt} = k \cdot P(N-P) \qquad P(0) = P_0$$

with simple growth





does log. growth predict level off?

Ex Suppose we ~~have~~ a herd of rabbits with a limiting population of say 100. Suppose initially we ~~have~~ 10 rabbits and after 7 days 20 rabbit. Find and solve a DE for the population if logistic growth is assumed

Model $\quad \frac{dP}{dt} = kP(100-P) \qquad P(0) = 10 \quad P(7) = 20$

Separable ODE

$$\frac{dp}{p(100-p)} = k\,dt$$

$$\frac{1}{100}\left(\frac{1}{p} + \frac{1}{100-p}\right)dp = k\,dt$$

$$\left(\frac{1}{p} + \frac{1}{100-p}\right)dp = 100k\,dt \quad \text{let } \bar{k} = 100k$$

$$\int \quad \ln|p| - \ln|100-p| = \bar{k}t + \ln c$$

$$\frac{p}{100-p} = c\,e^{\bar{k}t}$$

Now bring in condition

(1) $P(0) = 10 \Rightarrow \dfrac{10}{100-10} = c\,e^{0} \Rightarrow c = \dfrac{10}{90} = \dfrac{1}{9}$

So $\dfrac{p}{100-p} = \dfrac{1}{9}e^{\bar{k}t}$

(2) $P(7) = 20$ so $\dfrac{20}{100-20} = \dfrac{1}{9}e^{7\bar{k}}$

$$e^{7k} = \frac{9(2\cancel{0})}{8\cancel{0}} = 9/4 =$$

$$k = \frac{1}{7}\ln\frac{9}{4} = .1158$$

$$\text{so} \quad \frac{P}{100-P} = \frac{1}{9}e^{.1158t}$$

$$9P = e^{.1158t}(100-P)$$

$$= 100e^{.1158t} - e^{.1158t}P$$

$$(9+e^{.1158t})P = 100e^{.1158t}$$

$$\Rightarrow \quad P = \frac{100e^{.1158t}}{9+e^{.1158t}}$$

After 30 days $\quad P(30) = \dfrac{100e^{.1158(30)}}{9+e^{.1158(30)}} = 78.214$

so 78

Is this realistic?

(1) Assume no death

$$\frac{dP}{dt} = kP(100-P) + h$$

h - harvesting rate

Will this change things?

or $$\frac{dP}{dt} = kP(100-P) \cancel{+} - hP$$

<u>Gompertz Modeling</u> (possible project)

$$\frac{dP}{dt} = kP\log\left(\frac{P}{N}\right)$$ N - population limit

How does this compare w/ logistic growth

Two Populations – Competing Species

Foxes & Rabbits

Let F – Foxes, R – Rabbits

$\frac{dF}{dt} = -aF$, decay $\frac{dR}{dt} = bR$ growth

Now let them interact

$\frac{dF}{dt} = -aF + bR$, $\frac{dR}{dt} = cR - dF$, a, b, c, d

Can we solve this system of 2 eq^n's

for 2 unknowns

Epidemic Models SIR

S – susceptible

I – infected

R – removed

$$\frac{dS}{dt} = -\alpha \underline{IS}$$

$$\frac{dI}{dt} = \alpha \underline{IS} - \beta I$$

$$\frac{dR}{dt} = \beta I$$

← a nonlinear model