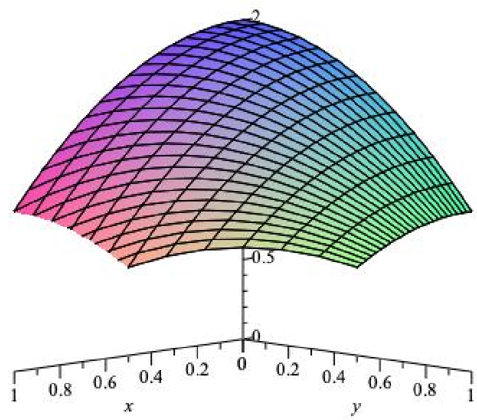
Calculus 3 - Volumes extra examples

Extra examples using double integrals to find the volume under a surface



where

$$V = \iint_R f(x, y) dA \tag{1}$$

where $dA = dxdy$ or $dydx$.

Example 1. pg. 987, #24 Find the volume under the plane $z = 4 - x - y$ on the region bound by $y = x$, $x = 0$ and $y = 2$

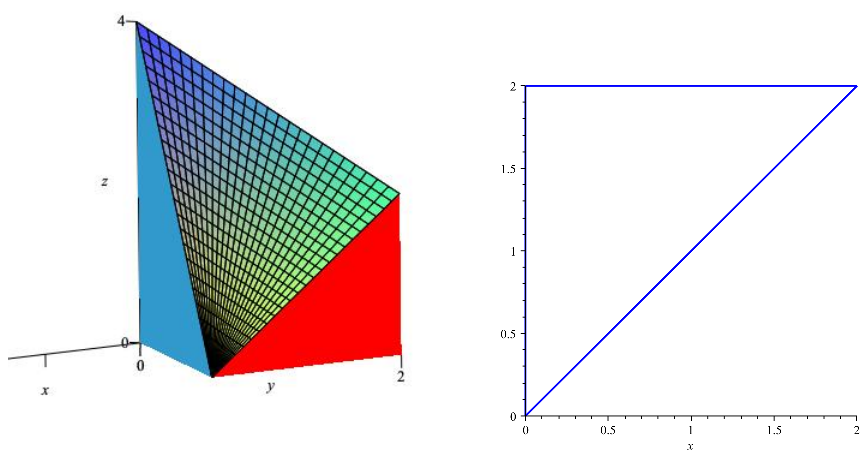


Figure 1: Region of integration

$$V = \int_0^2 \int_x^2 (4 - x - y) dydx$$

or

$$V = \int_0^2 \int_0^y (4 - x - y) dx dy \tag{2}$$

Example 2. Find the volume under the paraboloid $z = 2 - x^2 - y^2$ and inside the cylinder $x^2 + y^2 = 1$, for $z \geq 0$

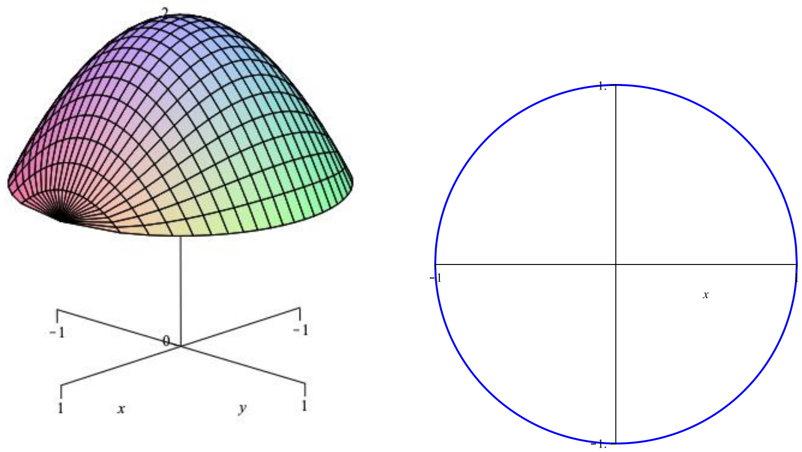


Figure 2: Region of integration

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 - x^2 - y^2) dy dx \tag{3}$$

or

$$V = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (2 - x^2 - y^2) dx dy \tag{4}$$

Due to symmetry

$$
\begin{aligned}
V &= 4\int_0^1 \int_0^{\sqrt{1-x^2}} (2 - x^2 - y^2) dy dx \\
&= 4\int_0^1 2y - x^2 y - \frac{2}{3}y^3\Big|_0^{\sqrt{1-x^2}} dx \\
&= 4\int_0^1 2\sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{2}{3}\left(\sqrt{1-x^2}\right)^3 dx \;\; (trig\ sub\ x = \sin\theta) \\
&= \frac{3\pi}{2}
\end{aligned}
\tag{5}
$$

Polar Coordinates

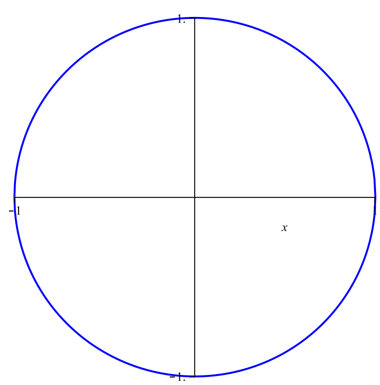
In calculus 2 we introduced polar coordinates where

$$x = r\cos\theta, \quad y = r\sin\theta \tag{6}$$

and

$$x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x} \tag{7}$$

Let's talk about sweeping out the region



We see that

$$r = 0 \to 1, \quad \theta = 0 \to 2\pi \tag{8}$$

so what about this integral from the last example

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2 - x^2 - y^2)dydx. \tag{9}$$

How would it change if we used r and θ instead of x and y? Would it becomes easier?

What about the volume under the half sphere $z = \sqrt{1 - x^2 - y^2}$

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1 - x^2 - y^2}\, dydx. \tag{10}$$