Inverse Method for Function Estimation in Turbulent DPHE Flow

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ABSTRACT-The article investigates the applicability of inverse methods in estimating the best values to be assigned to certain parameters which appear in turbulent flow through DPHE. The normalized study domain has been derived from original DPHE. Conjugate gradient method (CGM) technique is applied in the estimation of unknown heat transfer rate. No information is available for the unknown heat transfer rate and hence the procedure is classified as estimation of unknown quantity using simulated temperature measurement. The parameter optimization relies on the availability of data for the velocity profile, friction factor, viscosity and other turbulent parameters. It is shown that using these optimized parameters; it is possible to predict with more the parameter profile within the domain. The Prandtl mixing length theory model used to solve the turbulent energy equation for parameter estimation. Results shows that there is excellent estimation of heat transfer rate for the test case analyzed in this study domain of Double Pipe Heat Exchanger.

Keywords - *Inverse method, parameter estimation, turbulent flow, CGM*

1. INTRODUCTION

Inverting the energy equation is a problem of great interest in the sciences and engineering, in particular for modeling and monitoring applications [2]. In this work, we look at the energy equation depends on some domain containing a point heat source at known location. The magnitude of the heat source is assumed to be unknown and vary with time. This is very important utility in applications because it useful for determining the temperature of a body at the points where direct measurement is infeasible or inaccurate or difficult to measure. For example, when a space vehicle reenters the atmosphere, the temperature experienced by the heat shield can be too large for traditional sensors, and thus we must take measurements from distance and infer the true temperature [6]. The energy equation we consider is the linear heat equation and thus, the resulting optimization is linear function. A more general model of heat transfer, however, involves the nonlinear energy equation, which results in a nonlinear optimization problem. We are interested in the solution of such nonlinear problems from an inverse theory point of view, but the numerical solution of nonlinear partial differential equations is a complicated topic, discussion of which is outside the scope of this work. Thus, we stick to the linear energy equation while exploring algorithms that will work in the general case.

Unfortunately, the inverse problem is ill-posed which makes numerical solution difficult without the introduction of regularization or other analysis tools [4]. In particular, we will look at the effects of noise on our reconstruction both with and without a form of Tikhonov-type regularization parameter to promote smoothness of the solution in time.

2. PHYSICAL SET UP

2.1 GEOMETRY AND NORMALIZATION

Considering two-dimensional cylindrical domain that covers the annular region between two concentric circles. We consider hydro-dynamically developed, thermally developing turbulent forced convection of a constant property fluid flowing inside an annulus of double pipe heat exchanger as shown in figure 1.

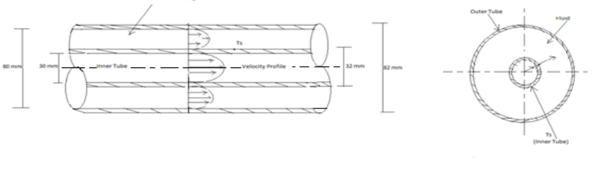
r1= Inner Pipe Radius, r2=Outer Pipe Radius

The curvilinear coordinates (r,θ) vary on the intervals [r_1, r_2] and [0, 2π] respectively. These curvilinear coordinates are related to cartesian coordinates (x, y) by transformation equations

$$x(r,\theta) = r\cos\theta,\ y(r,\theta) = r\sin\theta$$

The inverse transformation is given by

$$r(x,y) = \sqrt{x^2+y^2},\ \theta(x,y) = \tan^{-1}\frac{y}{x}$$



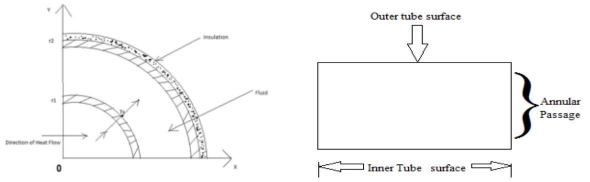


Figure- 1 Velocity Profile, Cross-Section and 2-D Convective Heat Flow in DPHE

We take, one of the curvilinear coordinates 'r' is constant on each of physical boundaries, while the other coordinates 'θ' varies monotonically over the same range around each of the boundaries. The system can be represented as a rectangle on which the two physical boundaries correspond to the top and bottom sides(figure-2):

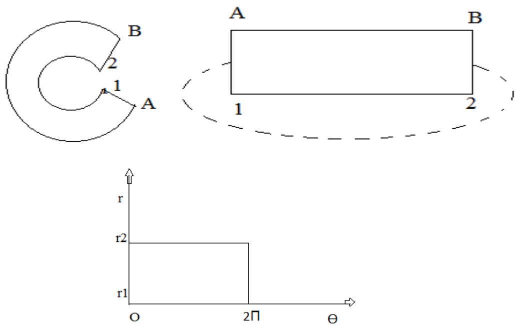


Figure- 2 Conversion of Cylindrical Coordinate System to Rectangular System

Coding point of view the system transformed in to rectangular coordunates where the curvilinear coordinates, r and θ are independent Variables.

The curvilinear coordinates (r, θ) can be normalized to the interval [0, 1] by introducing the new curvilinear coordinates (ζ, η), where,

$$\zeta = \theta / 2\pi , \ \eta = \frac{r - r_1}{r_2 - r_1}$$

$$\theta(\zeta) = 2\pi\zeta , \qquad r(\eta) = r_1 + (r_2 - r_1)\zeta$$

The transformation then may be written as under

$x(\zeta , \eta) = [r_1 + (r_2 - r_1)\eta] \cos(2\pi\zeta)$ and $y(\zeta , \eta) = [r_1 + (r_2 - r_1)\eta] \sin(2\pi\zeta)$

Where, both ζ and η are varying on the interval [0, 1]. This is a mapping of annular region between the two circles in the physical space onto the unit square in the transformed space, i.e., each point (x, y) on the annulus corresponds to one, and only one point (ζ , η) on the unit square:

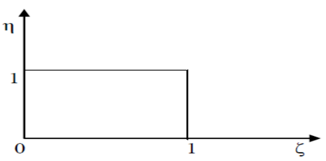


Figure- 3 Normalized Form of Curvilinear Coordinate System

The bottom (η = 0) and top (η = 1) of the square correspond respectively to the inner and outer circles, $r = r_1$ and $r = r_2$. The sides of square, ζ = 0 and ζ = 1 correspond to θ = 0 and θ = 2π, respectively as shown in figure-4.

The solution function T(x, y) of this differential equation describes the temperature, for example, a thin metal plate of unit area, at every position(x, y), $0 \le x, \ y \le 1$, at any time t ≥ 0. At the edge points of the plate, we have constant temperatures at x=0, x=1, y=0 and y=1. At time t = 0, the temperature of every point (x, y) is given by T(x, y). In the simplest case, we may assume that T(x, y) is a constant function with its value somehow in the range defined by the boundary conditions. For example, the temperature at position (x = 0.4, y =0.5) at time t = 5 is given by T(5, 0.4, 0.5). We try to approximate the values of the solution function T (t, x, y).

2.2 THE INVERSE PROBLEM

The theoretical model is presented which solves the inverse problem of heat transfer through convection to accurately estimate the heat transfer function varying at time of the thermal boundaries of the annulus. This approximate determination is based on measurement of transient temperature captured by a thermocouple on the heated surface. The conjugate gradient method with adjoint problem for function estimation is used to determine the heat transfer rate approximately.

In contrast to the forward problem of Equation (1), we will not assume knowledge of heat sources inside the domain - at least, not total knowledge. Rather, we can initially assume the domain of interest that has a uniformly at heat profile without loss of generality to be u(x; 0) = 0. According to Neumann boundary conditions and in lieu of any sources or sinks the solution of Equation (1.1) is simply T(r; t) = 0 for all time.

For our setup, following Ozisik and Orlande in [6], we are interested in the case of a single source q_s at a known location in the domain, r_s, which has some time-varying strength, i.e.,

$$q_s(r, t) = f(t)\delta(r - r_s)$$

Where, $\delta(r)$ is the Direc delta function-

$$\boldsymbol{\delta(r)} = \begin{cases} \infty & \boldsymbol{r = 0}, \\ \boldsymbol{r} & \boldsymbol{else.} \end{cases}$$

Knowing the initial heat profile we could use Equation (1.1) to predict the effect of r_s on the temperature distribution T - if we knew f (t). Instead, however, we will look at the inverse problem: given measurements of T at discrete points in space and time, recover f.

Formally, we consider a number N_r of receivers located at points in the domain distinct from r_s. Using $r_{r,i}$ to denote the position of the i-th receiver, we assume we have accessed to measurements of the temperature T at each point $r_{r,i}$ for N_t different times between 0 and t_f . The problem of interest is the function estimation problem on some domain with boundary:

The mathematical formulation for this steady-state forced convection problem is given as follow for the domain shown in figure 4.

$$k\frac{\partial^2 T}{\partial y^2} = u(y)\rho C_p \frac{\partial T}{\partial t} \qquad in\ 0 < y < L, 0 < x < b, t > 0 \tag{1.1}$$

$$k\frac{\partial T}{\partial y} = f(t) =? \qquad at\ y = 0\ for\ 0 < x < b, 0 \le t \le t_f \tag{1.2}$$

$$k\frac{\partial T}{\partial y} = f(t) =? \qquad at\ x = 0\ for\ 0 < y < L, 0 \le t \le t_f \tag{1.3}$$

$$k\frac{\partial T}{\partial y} = 0 \qquad at\ y = L\ for\ 0 < x < b, t > 0 \tag{1.4}$$

$$k\frac{\partial T}{\partial y} = 0 \qquad at\ x = b\ for\ 0 < y < L, t > 0 \tag{1.5}$$

$$T = T_L \qquad at\ y = L\ for\ 0 < x < b, t > 0 \tag{1.6}$$

$$T = F(x) \qquad at\ y = 0\ in\ 0 < x < b, t > 0 \tag{1.7}$$

The Conjugate Gradient Method (CGM) based on Adjoint Problem for Estimation of the Function (Technique –IV) is used to model the inverse problem. The function *f(t)* is regarded to be unknown.

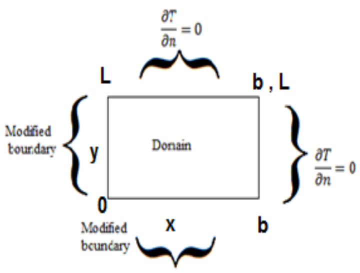


Figure- 4 Domain for Inverse Convective Heat Transfer

In order to find f(t). We refer to the above as the inverse problem. However we note that the assumption which we can measure $T(r_{r;i};\ t_j)$ exactly is a stronger assumption than one should be willing to make, given that sensor noise / measurement error is always an issue in real-world problems. As such, we will consider the case where we are not given exactly $T(r_{r;i};\ t_j)$, but, rather, measurements subject to white noise with standard deviation σ. We will see that the impact of this noise on our solution can be quite dramatic and affects our solution algorithm.

2.3 NON LINEAR CONJUGATE GRADIENT

To solve the inverse problem of equation 1, we can use the non linear conjugate gradient method. In essence, given an initial guess, the conjugate gradient method minimizes a function J(x) by, at each iteration, choosing a new guess by taking the old guess and tacking on an additional term that pushes the solution closer to the optimal one [7]. In contrast to the method of gradient descent the new direction is not required to be strictly in the direction of the gradient, which means that subsequent directions are not constrained to be orthogonal.

The basic idea behind the algorithm is [6], given initial guess $f_k = f_0$,

- To solve the temperatures, $T_k(r_{r;i};\ t_j)$ that result from f_k at the measurement points using the forward problem.
- To Calculate the residual error, i.e., the difference between the resultant temperatures, $T_k(r_{r;i};\ t_j)$; and the measured temperatures, $T_{i;j}$.
- To find a search direction by inverting the heat equation to determine what heat source function would account for the residual error. Potentially modify this search direction by taking into account other known or assumed information about the heat source function.
- Adjust your guess fk by adding on some component in the new search direction to form fk+1.
- Set k = k + 1 and repeat.

Mathematically, we define $T_{meas} = (T_{i;j})$ to be the vector of all measured temperatures over all times, and $T_f \equiv H_f = \{(r_i;\ t_j\ ;\ f\ )\}$ to be the vector of calculated temperatures at all measurement locations and times using the source f , where H is the operator that corresponds to solving the forward problem and vectorizing appropriately. The difference is that the first is strictly physical and results from actual measurements and the second is functional and results from computation. We are looking at a concrete, least-squares minimization problem: we want the source $\hat{f}$ that satisfies

$$\hat{f} = \underset{f}{argmin}\ J[f]$$

$$J[f] = \frac{1}{2}\left\|Tf - Tmeas\right\|_2^2$$

Above, we have adopted the convention of writing the source as a vector rather than a continuous function of time, as we note that any numerical computations will be limited to the discrete setting. We note that the resolution of f may be increased without difficulty (except perhaps computational cost) and thus this in no way limits the applicability of our theory. We are also highly interested in regularization of the inverse problem, where we can employ information known already about f in some meaningful way. The regularization in which we are interested is Tikhonov approach to promote smoothness of the solution $\hat{f}$. In particular, if we define D to be a finite difference matrix that calculates the discrete first derivative, we can modify the cost function in above Equation to a regularized cost function.

$$J_r[f] = \frac{\left\|T_f - T_{meas}\right\|_2^2 + \lambda\left\|Df\right\|_2^2}{2}$$

In the above λ is the regularization parameter that represents the trade-off between the data and the a priori information in general, choosing its value is a difficult task and it frequently must be hand-tuned. The nonlinear conjugate gradient method, in a more precise algorithmic form than the above, can be seen in Algorithm 1. The variable is known as the conjugation coefficient and represents the fact that we want to move in the improved direction, but not so far that the algorithm becomes unstable or oscillates, and β represents the fact that we want to be a completely memory-less algorithm like gradient descent, but to keep the track of the past directions and continue to use that information. Beyond γ and λ, Algorithm 1 requires the computation of the gradient of J at each iteration. We below discuss the theory and method behind computing these values.

2.4 THE GRADIENT

To compute the gradient of the cost function, $\nabla J[f]$, we will first look at the case without regularization, Equation (2).

Algorithm 1

$k = 0$

$f^{(k)} = f_0$

$d^{(k)} = -\nabla J[f^{(k)}]$

$\boldsymbol{repeat}$

$\quad Compute\ \gamma^{(k)}$

$\quad f^{(k+1)} = d^{(k)} + \gamma^{(k)} f^{(k)}$

$\quad Compute\ \beta^{(k)}$

$\quad d^{(k+1)} = -\nabla J\left[f^{(k+1)}\right] + \beta^{(k)} d^{(k)}$

$\quad k = k + 1$

$\boldsymbol{until}\ Convergence$

$\hat{f} = f^{(k)}$

Cued by Jarny and Ozisik in [3], we note that the gradient is related to the directional derivative by the inner-product,

$$\nabla_{\Delta f} J[f] = \langle \nabla J[f], \Delta f \rangle$$

Where Δf is some direction in which f is to be perturbed, An alternative expression which we can use for the directional derivative is simply its definition,

$$\nabla_{\Delta f} J[f] = \lim_{e \to 0} \frac{J[f + \varepsilon \Delta f] - J[f]}{\varepsilon}$$

Simple algebra allows us to use the definition to calculate the directional derivative directly,

$$\nabla_{\Delta f} J[f] = \lim_{e \to 0} \frac{\|H(f + \varepsilon \Delta f) - T_{meas}\|_2^2 - \|Hf - T_{meas}\|_2^2}{2\varepsilon}$$

$$= \langle H^T (Hf - T_{meas}), \Delta f \rangle$$

By equating the above with Equation (4) and noting that the choice of direction was arbitrary, we can assert that

$$\nabla J[f] = H^T (Hf - T_{meas})$$

for the case of no regularization. Because the gradient is additive, it is simple to see that adding regularization as in Equation (3) yields

$\nabla J_r[f]$	=	$\nabla J[f] + \lambda \nabla \left(\frac{\|Df\|_2^2}{2} \right)$
	=	$H^T (Hf - T_{meas}) + \lambda D^T D f$

The problem now becomes determining what is meant by the adjoint operators H^T and D^T. As D is the time-derivative operator, it is easy to show that the adjoint operator is the negative time-derivative, i.e., $D^T = - D$.

The application of the adjoint heat equation operator (with measurement), H^T, is more complicated, and a derivation is outside the scope of this project. We below give the statement of the operator H^T without proof.

2.5 THE ADJOINT

In [3], Jarny and Ozisik describe the adjoint problem stated here. As H is an operator that takes source-function space to measurement space, H^T should, naturally, be a function that takes measurement space to source-function space. The final form of the adjoint operator involves the solution to the adjoint problem,

$$-\frac{\partial \psi(r,t)}{\partial t} = \nabla^2 \psi(r,t) + \sum_{i=1}^{N_r} \overline{T}_i(t) \delta(r - r_{r,i}), \quad (r,t) \in \Omega \times (0, t_f) \qquad (6)$$

$$\frac{\partial \psi(r,t)}{\partial t} = 0, \qquad (r,t) \in \Omega \times (0, t_f)$$

$$\psi(r, t_f) = 0, \qquad (4)$$

where $\overline{T}_i(t)$ is the continuous function achieved by time-interpolation of the measurements from sensor i. We note that Equation (12) is a final-value problem, but can be converted to an initial-value problem by making time go backwards [3].

The definition of H^T is simply derived as,

$$H^T T = \psi,$$

where ψ is given by solving the adjoint problem with the temperature information in T broken apart once again into separate sensors and then interpolated in time.

2.6 THE CONJUGATION COEFFICIENT

There are several heuristics for calculating the conjugation coefficient, γ, including, but not limited to, the Fletcher-Reeves method, and the Polak-Ribiere method [4]. We use the latter the expression for which is

$$\gamma^{(k)} = \frac{-\langle \nabla J\left[f^{(k+1)}\right], \nabla J\left[f^{(k+1)}\right] - \nabla J[f^{(k)}] \rangle}{\|\nabla J[f^{(k)}]\|_2^2}$$

2.7 THE STEP SIZE

Combined γ and ∇J are all we need to determine the next search direction. The next variable to be determined is the optimal step size in that direction, β.

To determine the optimal β for the step size, we need to calculate how the output is perturbed as a response to a perturbation in the input of the forward problem of Equation (1.1). Thus, we are interested in the following sensitivity problem [3],

$$\frac{\partial \Delta T(r,t)}{\partial t} = \nabla^2 \Delta T(r,t) + \Delta f(t)\delta(r-r_s), \quad (r,t) \in \Omega \times (0,t_f) \qquad (8)$$

$$\frac{\partial \Delta T(r,t)}{\partial t} = 0, \quad (r,t) \in \Omega \times (0,t_f)$$

$$\Delta T(r,t_f) = 0,$$

Using this to solve for ΔT givenΔf , we can that compute $\beta^{(k)}$ by differentiating $J\left[f^{(k+1)}\right]$with respect to β and equating that to zero. This yields [6]

$$\beta^{(k)} = \frac{\langle T_f^{(k)} - T_{meas,}\Delta T^{(k)}\rangle}{\|\Delta T^{(k)}\|_2^2} \qquad (9)$$

where, once again, $T_f^{(k)}$ is the vectorized measurements from all sensors across time.

3. TURBULENCE MODELLING IN INVERSE PROBLEM

3.1 Eddy-Viscosity Models -Eddy- Viscosity Hypothesis

The mean shear stress has both viscous and turbulent parts. In simple shear:

$$\tau = \mu \frac{\partial U}{\partial y} - \rho \overline{uv} \qquad (10)$$

In above equation the first component is viscous and the second component is turbulent [8, 9, 10]. The most popular type of turbulence model is eddy viscosity model (EVM) which assumes that turbulent stress is proportional to mean velocity gradient in a manner similar to viscous stress. In simple shear:

$$\mu_t \frac{\partial U}{\partial y} = -\rho \overline{uv} \qquad (11)$$

μ_t is called an eddy viscosity or turbulent viscosity. This mean shear stress is then,

$$\tau = \mu_{eff} \frac{\partial U}{\partial y} \qquad (12)$$

Where, the total effective viscosity is

$$\mu_{eff} = \mu + \mu_t \qquad (13)$$

with the eddy-viscosity hypothesis, closure of the mean-flow equations now rests solely on the specification of μ_t, a property of the turbulent flow.

3.2 Mixing Length Models (Prandtl,1925)

Eddy viscosity:

$$\mu_t = \rho\, \nu_t \qquad where\ \nu_t = u_0\, l_m$$

The mixing length l_m is specified algebraically and the velocity scale u_0 is then determined from the mean-velocity gradient. In simple shear:

$$u_0 = l_m \left|\frac{\partial U}{\partial y}\right|$$

The model is based on the premise that if a turbulent eddy displaces fluid particle by distance l_m its velocity will differ from its surrounding by an amount $l_m|\partial U/\partial y|$.

$$\mu_t = \rho \times \nu_t = \rho l_m^2 \left|\frac{\partial U}{\partial y}\right|$$

4. NUMERICAL RESULTS

The flow considered in the annulus is hydrodynamically developed and the section of interest is adequately far from the inlet section, so that the velocity profile becomes unvarying with the axial direction. Due to the symmetry nature only one fourth cross section of the flow geometry is modeled. In the two dimensional environment the velocity is increasing in the positive vertical direction and constant velocity values in positive horizontal direction. In annulus radial cross-sectional form and in x-y directions have been considered and the temperature strength has been computed. The viscous dissipation is examined as per the geometry of annuli shown in figure 1. The function estimation with the customized boundaries considering the heat flux applied on the boundary of the domain which is the realistic case. This is the core work behind the development of this code which can resemble the actual heat transfer process in the double pipe heat exchanger. In the heat exchanging process the heat is applied to the one medium to the other medium at the contact surface between two medium. In order to incorporate the application of heat the boundary conditions are modified in the code.

The MATLAB code is tested with the given true function. We use a simple finite-difference discretization to solve the forward problems. In particular, we use an implicit discretization in time and a simple central difference in space, enforcing Neumann conditions at the spatial boundaries by assuming ghost points outside the domain. The next task is to incorporate the boundary conditions. The code solves the heat convection problem with Neumann boundary condition. The requirement as per the real situation is that the heat flux is specified at the particular boundary where the heat transfer is taken place. These boundary conditions are modified in the code. Maximum allowable iterations are 200 and Minimum required reduction in residual per iteration is 0.1%. Tolerable residual size is 1E-12.

The Function estimation of inverse heat convection problem is done and compared with the inverse heat convection problem in turbulent flow. The force heat convection problem predicted without turbulence model is impossible. Prandtl mixing length theory model has been used and compared with smooth tube without turbulence model in IHTP MATLAB code. Looking to large domain and actual size of the annular duct heat exchanger a small section is considered for analysis within which the turbulent viscosity has been calculated. The turbulent viscosity has been considered constant for a whole domain which is not realistic. The error plot and temperature contour plot have been compared for two cases. The cases are as under:

1. Inverse Laminar Heat Convection Problem
2. Inverse Turbulent Heat Convection Problem with Prandtl mixing length theory model

Figure-5 -6 shows the comparison of estimation of the function with the absolute error to that of actual function with modified boundaries for above cases. This shows that the result is in good agreement. The closer look at the result depict that each time the direct problem is solve and the same temperature is measure at the measurement location. The code estimates the function with regularization.

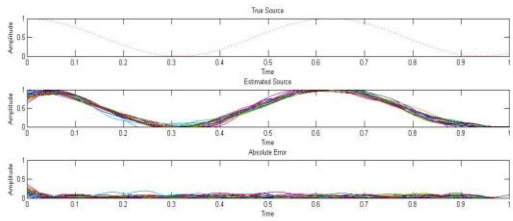


Figure - 5 Two dimensional heat convection problem with noise, regularization with α = 1E-3 and specified boundary conditions for tube without adding turbulent viscosity

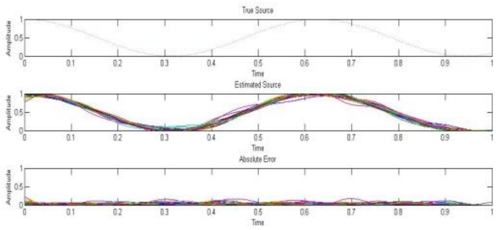


Figure - 6 Two dimensional heat convection problem with noise, regularization with α = 1E-3 and specified boundary conditions for tube with turbulent viscosity and Prandtl Mixing Length Turbulence Model

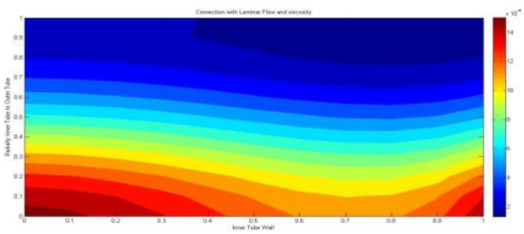


Figure-7 Temperature Contours of 2D Problem with noise, regularization with α = 1E-3 and specified boundary conditions without Turbulent Viscosity

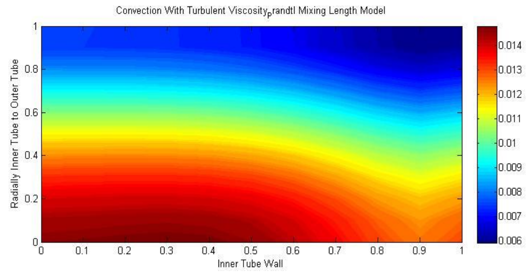


Figure-8 Temperature Contours of 2D Problem with noise, regularization α = 1E-3 and specified boundary conditions with turbulent viscosity and Prandtl Mixing Length Turbulence Model

Figure- 7 & 8 shows comparison of the temperature contours of the convection process which reflect the effect of boundary conditions for above cases. The heat is convected from the bottom and left side boundary in the rest of the computational domain.

CONCLUSION

The concluded results for the present numerical investigation with insert for Double Pipe Heat Exchanger are as follows:

1. Adjoint based conjugate gradient method is used to solve the Inverse Problem to overcome the difficulties with well-posed problem in measurement for DPHE. The numerical solution to the heat convection problem is obtained using the concept of inverse heat transfer. No prior information of the function of the quantities is required in the inverse analysis in the conjugate gradient technique. The "boundary conditions" are customized to bear a resemblance to the real life problem of interest.

2. The results attained finally exhibit little noise and the algorithm proven to be robust in several trials. The unsteady function dependent on time is estimated satisfactory. The contours of temperature confirm the turbulent convection process consistently in the focused study domain. The Prandtl Mixing Length turbulence Model is used to capture the turbulence in fluid flow.

3. In real life there are number of parameters that affect the heat transfer process. The actual heat transfer coefficient always varies due to the ongoing unavoidable physical phenomenon like corrosion, erosion, cavitations, flow patterns, variation in temperature of fluid, environmental changes etc. And it is not possible to measure the exact heat transfer coefficient directly without measuring the temperature of the complex configuration of heat exchanger equipments. This method is capable of estimating the time varying strength of the unknown function.

4. Using this proposed theoretical model accurate estimation of heat transfer rate becomes possible and the temperature distribution in the convective system can be calculated. Based on the other selection criteria such as temperature difference, heat transfer surface and with the estimated amount of heat transfer, the heat transfer coefficient can be calculated. Thus estimation of the time varying heat transfer rate by inverse method is a rational alternative towards the performance assessment of DPHE.

5.

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