Math 4381/6378 Symmetry

Last class we found the symmetries of

$$u_t = u_x^2$$

to be

$$T = c_1 x^2 + (2c_2 t + c_3)x + 4c_4 t^2 + c_5 t + c_6$$

$$X = -2(2c_1 x + 2c_2 t + c_3)u + c_2 x^2 + (4c_4 t + c_7)x + 2c_8 t + c_9$$

$$U = -4c_1 u^2 + (2c_2 x - c_5 + 2c_7)u - c_4 x^2 - c_8 x + c_{10}$$

Let consider an example on how to use these. Let choose $c_6 = 1$, $c_8 = 1$ other $= 0$

so $T = 1$, $X = -2t$ $U = x$

we solve

$$r_t - 2t r_x + x r_u = 0$$

$$s_t - 2t s_x + x s_u = 1$$

$$V_t - 2t V_x + x V_u = 0$$

By MofC

$$\frac{dt}{1} = \frac{dx}{-2t} = \frac{du}{x}\ ;\ dr = 0$$

$$c_1 = x+t^2, \quad du = x\,dt = (c_1 - t^2)\,dt$$

$$u = c_1 t - \frac{t^3}{3} + c_2$$

$$= (x+t^2)t - \frac{t^3}{3} + c_2$$

$$= xt + \frac{2}{3}t^3 + c_2$$

so $\cancel{t}r = R(x+t^2,\ xt + \frac{2}{3}t^3 - u)$

$$s = t + S(x+t^2,\ xt + \frac{2}{3}t^3 - u)$$

$$v = V(x+t^2,\ xt + \frac{2}{3}t^3 - u)$$

we choose

$$r = x+t^2, \quad s = t, \quad v = u - xt - \frac{2}{3}t^3$$

or $\quad t = s, \quad x = r - s^2 \quad u = v + rs - \frac{1}{3}s^3$

Now we change variables and derivatives

Way 1 Standard chain Rule

$$u_t = u_r r_t + u_s s_t = 2t u_r + u_s = 2s u_r + u_s$$

$$= 2s \frac{\partial}{\partial r}\left(v + rs - \frac{1}{3}s^3\right) + \frac{\partial}{\partial s}\left(v + rs - \frac{1}{3}s^3\right)$$

$$= 2s\left(v_r + s\right) + v_s + r - s^2$$

$$= 2s v_r + v_s + r + s^2$$

$$u_x = u_r r_x + u_s s_x = u_r = v_r + s$$

Way 2 Jacobians

$$u_t = \frac{\partial(u,x)}{\partial(t,x)} = \frac{\partial(u,x)}{\partial(v,s)} \Big/ \frac{\partial(t,x)}{\partial(v,s)}$$

$$= \frac{\partial\left(v + rs - \frac{1}{3}s^3,\ r - s^2\right)}{\partial(r,s)} \Big/ \frac{\partial(s,\ r - s^2)}{\partial(v,s)}$$

$$= \begin{vmatrix} v_r + s & v_s + r - s^2 \\ 1 & -2s \end{vmatrix} \Big/ \begin{vmatrix} 0 & 1 \\ 1 & -2s \end{vmatrix}$$

$$= 2s v_r + v_s + r + s^2$$

$$u_x = \frac{\partial(t,u)}{\partial(v,s)} \Big/ -1 = -\begin{vmatrix} 0 & 1 \\ v_r + s & v_s + r - s^2 \end{vmatrix} = v_r + s$$

Now we consider the original PDE

$$u_t = u_x^2 \Rightarrow 2sv_r + v_s + r + s^2 = (v_r + s)^2$$

so $2s\cancel{v_r} + v_s + r + \cancel{s^2} = v_r^2 + 2\cancel{s v_r} + \cancel{s^2}$

$$\Rightarrow \quad v_s + r = v_r^2$$

Now we assume $v = v(r)$ only $\quad$ so $\quad$ because

$$v_r = \pm\sqrt{r} \Rightarrow v = \pm \frac{2}{3} r^{3/2} + c$$

Now back substitute

$$u - xt - \frac{2}{3}t^3 = \pm\frac{2}{3}(x+t^2)^{3/2} + c$$

or $\quad u = xt + \frac{2}{3}t^3 \pm \frac{2}{3}(x+t^2)^{3/2} + c$

an exact solⁿ of $u_t = u_x^2$

Question - can we bypass the introduction of the new coordinate and get the solution form directly?

Now bring in #1

so $F_r\left(-\bar{U} r_u + r_u (T u_t + X u_x)\right)$

$$+ F_v\left(-\bar{U} v_u + v_u (T u_t + X u_x)\right) = 0$$

$$\Rightarrow \quad (T u_t + X u_x - \bar{U})(F_r r_u + F_v v_u) = 0$$

the 2nd term implies $\frac{\partial F}{\partial u} = 0 \Rightarrow F$ indep of $u$ 
mod.

so $T u_t + X u_x = \bar{U}$

Consider a sol$^n$

$$u = f(t, x)$$

and it's invariant under $\bar{t} = t + \varepsilon T$, $\bar{x} = x + \varepsilon X$, $\bar{u} = u + \varepsilon \bar{U}$

so $\bar{u} = f(\bar{t}, \bar{x}) \Rightarrow \quad u + \varepsilon \bar{U} = f(t + \varepsilon T, x + \varepsilon X)$

expend $\quad u + \varepsilon \bar{U} = f(t, x) + \epsilon (T f_t + X f_x) + O(\epsilon^2)$

so if $u = f(t, x)$ then to order $O(\epsilon^2)$

$$\bar{U} = T f_t + X f_x \quad \text{or} \quad T u_t + X u_x = \bar{U}$$

$$\because u = f$$

Given $T, X, \bar{U}$ the symmetries

we solve

$$T r_t + X r_x + \bar{U} r_u = 0$$
$$T s_t + X s_x + \bar{U} s_u = 1 \qquad (A)$$
$$T v_t + X v_x + \bar{U} v_u = 0$$

These we solve giving

$$r = R(A,B), \quad s = C + \tilde{S}(A,B) \quad v = V(A,B)$$

where $A, B, C$ are some fcts of $t, x, u$.

Then we assume $v = F(r)$ or $F(r,v) = 0$ (it may be implicit)

no $s$ (no $s$)

so $F_r(r_t + r_u u_t) + F_v(v_t + v_u u_t) = 0$

$F_r(r_x + r_u u_x) + F_v(v_x + v_u u_x) = 0$

mult 1st by $T$, 2nd by $X$ & add

$$F_r\left(T r_t + X r_x + r_u (T u_t + X u_x)\right)$$
$$+ F_v\left(T v_t + X v_x + v_u (T u_t + X u_x)\right) = 0$$

$T u_t + X u_x = U$ is called the invariant surface condition (ISC)

Ex we considered

$$u_t = u_x^2$$

and saw 1 sym $T = 1,\ X = -2t,\ U = x$

so ISC $\quad u_t - 2t u_x = x \quad$ looks kinda familiar

$$\frac{dt}{1} = \frac{dx}{-2t} = \frac{du}{x}$$

$$c_1 = x + t^2$$

$$du = x\,dt$$
$$= (c_1 - t^2)\,dt = c_1 t - \frac{t^3}{3} + c_2$$
$$u = (x + t^2)t + c_2 - t^3/3$$
$$= xt + \frac{2}{3}t^3 + c_2$$

so sol'n $\quad u = xt + \frac{2}{3}t^3 + f(x + t^2)$

$$u_t = x + 2t^2 + f' \cdot (2t), \quad u_x = t + f'$$

$$u_t = u_x^2 \Rightarrow x + 2t^2 + \cancel{2tf'} = (t + f')^2 = t^2 + \cancel{2tf'} + f'^2$$

$$f'^2 = x + t^2 \quad \text{if } r = x + t^2 \text{ we get } f'^2(r) = r$$

same ODE