Limits

The we first try and evaluate a limit

we do a direct substitution so

$$\lim_{x\to 1} 2x-1 = 2(1)-1 = 1$$

In general

$$\lim_{x\to a} c = c \qquad \lim_{x\to a} x = a \qquad \lim_{x\to a} x^n = a^n$$

Properties

If $\lim_{x\to a} f(x) = L$ $\lim_{x\to a} g(x) = K$

$$\lim_{x\to a} f \pm g = L \pm K \qquad \lim_{x\to a} cf = cL$$

$$\lim_{x\to a} f\cdot g = LK \qquad \lim_{x\to a} \frac{f}{g} = \frac{L}{K}, \; K \neq 0$$

$$\lim_{x\to a} f^n = L^n$$

Now we come to the analytical part of evaluating limits. The techniques are

(1) factoring
(2) rationalization
(3) squeeze th^m

the we do by example

(1) <u>factoring</u>

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x\to 1} x+1 = 2$$

<u>Note:</u> we can only cancel the $x-1$ term if $x \neq 1$ but a limit is about getting close to $x=1$ without being at $x=1$ so as long as we stay away from $x=1$ we are ok.

(2) Rationalization

multiply by conjugate of top

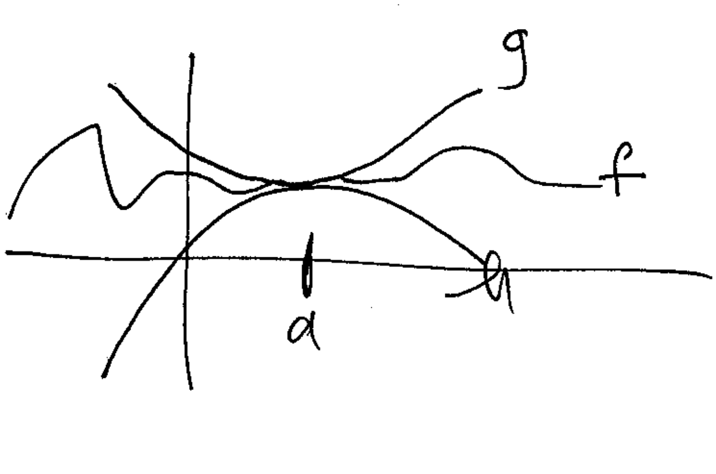
$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$$

$$\lim_{x\to 0} \frac{(x+\cancel{1}) + \cancel{\sqrt{x+1}} - \cancel{\sqrt{x+1}} - \cancel{1}}{x\left(\sqrt{x+1}+1\right)}$$

$$\lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}$$

$$\lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$ as we saw before

Squeeze Th^m



if $h \le f \le g$
near $x=a$

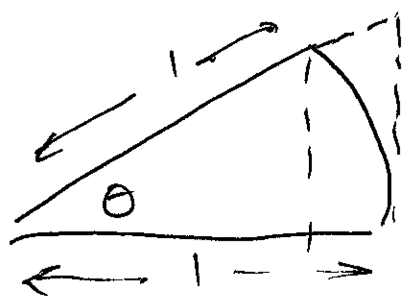
and $\lim_{x\to a} h(x) = L \qquad \lim_{x\to a} g(x) = L$

then $\lim_{x\to a} f(x) = L$

Here we'll use the squeeze thm to prove

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Consider a sector of a circle inside 2 triangles. Now the areas are

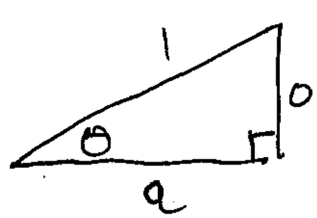


$$A_{\triangle} \le A_{\triangleleft} \le A_{\triangle}$$

we will assume that radius of the circle $= 1$

area smaller $\triangle$

Now from trig



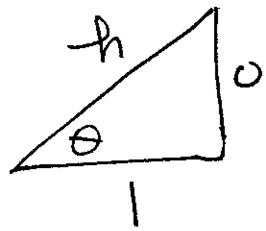
$\sin\theta = o$

$\cos\theta = a$

$$A = \frac{1}{2}\sin\theta\cos\theta$$

area sector $a = r\theta = \theta \quad \because r = 1$

area larger triangle



~~sin~~ $\tan\theta = \frac{o}{1}$

so $A = \frac{1}{2}(1)(o) = \frac{1}{2}\tan\theta$

so

$$\cancel{\tfrac{1}{2}}\sin\theta\cos\theta \le \cancel{\tfrac{1}{2}}\theta \le \cancel{\tfrac{1}{2}}\frac{\sin\theta}{\cos\theta}$$

Now divide by $\sin\theta$

$$\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta} \qquad \text{flip}$$

$$\frac{1}{\cos\theta} \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$

Now $\lim\limits_{\theta\to 0} \quad 1 \le \lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$

So by sq^z Th^m $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$