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Research Article

On Numerical Ranges of Convexoid Operators

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Abstract

Numerical range is useful in studying operators on Hilbert spaces. In particular, the geometrical properties of numerical range often provide useful information about algebraic and analytic properties of an operator. The theory of numerical range played a crucial role in the study of some algebraic structures especially in the non-associative context. The numerical range of an operator depends strongly upon the base field. Motivated by theoretical study and applications, researchers have considered different generalizations of numerical range. Numerical range of an operator T_1 may be a point, or a line segment containing none, one or all of its end points. Numerical range of another operator T_2 may be an open set, closed set or neither. In this paper, we give results of numerical range of convexoid operators. Let H be an infinite dimensional complex Hilbert space and $B(H)$ be algebra of all bounded linear operators on H. $T \in B(H)$ is said to be convexoid if the closure of the numerical range coincides with the convex hull of its spectrum. In this paper, we determine the numerical ranges of convexoid operators. We employ some results for convexoid operators due to Furuta and numerical ranges due to Shapiro, Furuta and Nakamoto, Mecheri and Okelo. Some properties of numerical ranges are also discussed.

Keywords: Numerical range; Convexoid operator; Numerical radius; Spectrum.

Introduction

The notion of numerical range was first introduced by [1]. In the sixties the concept of numerical range was extended to operators on general Banach spaces in [2] and [3]. Their definitions though different are used equivalently in most of the applications. Both definitions influenced further development of the theory and since then it has been studied by several mathematicians. Numerical range is useful in studying operators [4]. In particular, the geometrical properties of numerical range often provide useful information about algebraic and analytic properties of an operator. The theory of numerical range played a crucial role in the study of some algebraic structures especially in the non-associative context. The numerical range of an operator depends strongly upon the base field [5].

Motivated by theoretical study and applications, researchers have considered different generalizations of numerical range. Numerical range of an operator T_1 may be a point, or a line segment containing none, one or all of its end points. Numerical range of another operator T_2 may be an open set, closed set or neither [6]. In this paper, we give results of numerical range of convexoid operators. Let H be an infinite dimensional complex Hilbert space and $B(H)$ be algebra of all bounded linear operators on H. $T \in B(H)$ is said to be convexoid if the closure of the numerical range coincides with the convex hull of its spectrum[7]. The set of all convexoid operators on H is denoted by $CX(H), D(T)$ denotes the domain of T while $R(T)$ denotes the range of T and $W(T)$ is the numerical range of the operator T.

Preliminaries

In this section, we give the definition of some terms that are useful in our work.

Definition 2.1. [7, Definition 2.1.26] Numerical range $W(T)$, of an operator T is the subset of the complex number given by $W(T) = \{\langle Tx, x\rangle, x \in H, \|x\| = 1\}$.

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Definition 2.2. [3, Definition 2.4] Numerical radius $w(T)$, of an operator T on, is given by $w(T) = sup\{|\lambda|;\ \lambda \in W(T)\} = sup\{\langle Tx, x\rangle, \|x\| = 1\}$.

Definition 2.3. [6, Definition 2.8] Convexoid operator, $CX(H)$, is a bounded liner operator T on a complex Hilbert space H such that the closure of the numerical range coincides with the convex hull of its spectrum i.e $\overline{W(T)} = co\ \sigma(T)$.

Definition 2.4. [4, Chapter 9, section 73]Spectrum of T denoted by $\sigma(T) = \{\lambda: T - \lambda I \text{ is not invertible}\}$ is the complement of the resolvent set.

Materials and methods

For a successful completion of this paper, background knowledge of topology, functional analysis, the operator theory, especially normal operators, selfadjoint operators, hyponormal operators on a Hilbert space, numerical range and the spectrum of operators on a Hilbert space is vital. We have stated some known fundamental principles which shall be useful in our research. The methodology involved the use of known inequalities and techniques like Cauchy-Schwartz inequality, Minkowski's inequality, parallelogram law and the polarization identity. Lastly, we used the technical approach of tensor products in solving the stated problem. Also the methodology involved the use of known inequalities and techniques like the polarization identity [9].

Technical approach of derivatives of fuzzy sets to characterize the numerical range of convexoid operators was also used. Spectral theory of linear operators on Hilbert spaces is a pillar in several developments in mathematics, physics and quantum mechanics. Its concepts like the spectrum of a linear operator, eigenvalues and vectors, spectral radius, spectral integrals among others have useful applications in quantum mechanics, a reason why there is a lot of current research on these concepts and their generalizations. Spectral theory is described as a rich and important theory as it relates perfectly with other areas including measure and integration theory and theory of analytic functions. For this particular work, Schwarz inequality [8], polarization identity, inner product [10] and parallelogram law [11] were used as in the results.

Result and discussions

In this section, results on numerical ranges of convexoid operators are given. The main task is to determine the numerical range of convexoid operators. We start by looking at convexoid operators then some properties of numerical ranges.

Proposition 4.1. Let H be a complex Hilbert space and $T: H \to H$ be convexoid then the following properties hold:

i. T is linear
ii. T is bounded
iii. T is continuous.

Proof: (i). A linear operator T is an operator such that the domain $D(T)$, of T is a vector space and the range $R(T)$, lies in the vector space over the same field and for all $x, y \in D(T)$ and scalar α , so that $T(x + y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx$. Let $x, y \in H$ and $\alpha, \beta \in \mathbb{C}$. Since H is a Hilbert space, we can have an orthonormal basis $B = \{e_1, e_2\}$ such that $\alpha = e_1$ and $\beta = e_2$. Since a basis has linearly independent vectors then T must be linear and hence $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$. Indeed, to show that the operator $Ty = x^2D^2y - 2y$ is linear, the first condition is $T(f + g) = Tf + Tg$ for any functions f and g.

$T[f + g] = x^2D^2[f + g] - 2[f + g]$
$= x^2 \frac{d^2}{dx^2}[f + g] - 2f - 2g$
$= x^2\left[\frac{d^2}{dx^2}f + \frac{d^2}{dx^2}g\right] - 2f - 2g$
$= x^2D^2f + x^2D^2g - 2f - 2g$
$= \{x^2D^2f - 2f\} + \{x^2D^2g - 2g\}$
$= Tf + Tg.$

For the second condition, $T[cf] = cTf$.
$T[cf] = x^2D^2[cf] - 2[cf]$
$= x^2cD^2f - c2f$
$= c\{x^2D^2f - 2f\}$
$= cTf$. Thus is T linear.

(ii). Suppose T is linear from B into C. T is said to be bounded on $D(T)$ if and only if there is a finite number m such that $\|Tf\| \leq m\|f\|(f \in D(T))$. T ia said to be bounded if and only if also $D(T) = B$.

(iii). Suppose that T is a linear operator from B into C. Then if T is continuous at some point $f \in D(T)$, it is continuous. If (f_n) is any sequence in $D(T)$ with limit f, then $Tf_n \to Tf$. Now if (g_n) is a sequence in $D(T)$ with limit g also $g_n - g + f \to f$.

Then
$T(g_n - g + f) \to Tf$, from which it follows that $Tg_n - Tg + Tf \to Tf$, that is $Tg_n \to Tg$.
Proposition 4.2. Let $T \in CX(H)$ then T^n is convergent for all $n \in \mathbb{N}$. Moreover T is positive if $\exists \alpha \in \mathbb{C}$ such that $\|T\| \geq \alpha\|x\| \forall \in H$.
Proof: If $T^n \subseteq CX(H)$ and $x_n \subseteq T$ is bounded sequence with bound $M > 0$, then $T^n x_n$ has a convergent subsequence namely $T^n x_{nk} \to y$. For notational convenience we denote the subsequence (x_{nk}) by (x_n). For $n > m$, we have
$||T^n x_n - T^n x||^2 = \langle T^n(x_n - x), T^n(x_n - x)\rangle$
$= \langle T^n T^n(x_n - x), (x_n - x)\rangle$
$\leq ||T^n T^n(x_n - x)||||x_n - x||$
$\leq 2MT^n||(x_n - x)||$
Assume T is compact and let $(x_n)_{n \subseteq \mathbb{N}}$ be a bounded sequence. As it is bounded we can construct a weakly convergent subsequence, call it $(x_n)_n$ again in abuse of notation such that $x_n \to x$ weakly as $n \to \infty$. Now we want to show that $T^n x_n \to Tx_n$ strongly.
NB
$||T^n(x_n - x)||^2 = \langle T^n(x_n - x), T^n(x_n - x)\rangle = \langle (x_n - x)TT^n(x_n - x)\rangle$
$\leq \|x_n - x\|\|TT^n(x_n - x)\| \leq c\|TT^n(x_n - x)\|$,
where we used that a weak convergent sequence is bounded. It suffices to show that $TT^n(x_n - x) \to 0$ strongly as $n \to \infty$. For this, note that $TT^n(x_n - x) \to 0$ weakly as TT^n is continuous (in the strong and thus also in the weak topology). Since T is a compact operator and limits are unique we also now have that $TT^n(x_n - x) \to 0$ as $n \to \infty$ which conclude the proof.
For the numerical range we have the following; some of the properties are stated as below :

i. $W(\alpha I + \beta T) = \alpha + \beta W(T)$ for $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$.
ii. $W(T^*) = \{\bar{\lambda} : \lambda \in W(T)\}$.
iii. $W(U^*TU) = W(T)$, for any unitary U.
iv. $W(T)$ lies in a closed disc of radius $\|T\|$ centered at origin.
v. $W(T)$ contains all eigenvalues of T.
vi. $W(I) = \{1\}$.
vii. $W(T)$ is convex.

Next we look at some of the results on numerical ranges.
Proposition 4.3. Let $T \in CX(H)$ then $W(T)$ is nonempty. Moreover, $0 \in W(T)$.
Proof: T is bounded below but not onto then $\{0\} \neq (ranT)^{\perp} = kerT^*$, hence $0 \in W(T^*)$, and therefore $0 \in W(T)$. If $0 \in W(T)$, then $\langle Tf, f\rangle \in W(T)$ for every vector f in the unit ball and not only for the unit vectors. Reason if $\|f\| = 1$ and $0 \leq t \leq 1$ then by convexity $\langle T(tf), tf\rangle = t^2\langle Tf, f\rangle = t^2\langle Tf, f\rangle + (1 - t^2).0 \in W(T)$
. Then the quadratic form $f \longmapsto \langle Tf, f\rangle$ is weakly continuous on bounded sets. Indeed if $\{f_n\}$ is bounded, not weakly convergent to f, then $|\langle Tf_n, f_n\rangle - \langle Tf, f\rangle| \leq |\langle Tf_n, f_n\rangle - \langle Tf, f_n\rangle| + |\langle Tf, f_n\rangle - \langle Tf, f\rangle|$
The first summand tends to 0 because $f_n \to f$ weakly. Furthermore, if T is an operator and λ is a complex number then $|\lambda| = \|T\|$ and $\lambda \in W(T)$. If $\lambda = \langle Tf, f\rangle$ with $\|f\| = 1$ then , $\|T\| = |\lambda| = |\langle Tf, f\rangle| \leq \|Tf\|.\|f\| \leq \|T\|$ so that equality holds everywhere. By Schwarz inequality $Tf = \lambda_0 f$ for some λ_0 and this then implies $\lambda_0 = \lambda_0\langle f, f\rangle = \langle \lambda_0 f, f\rangle = \langle Tf, f\rangle = \lambda$, then λ is an eigenvalue of T. It follows that if λ is a number in $\overline{W(T)}$ such that λ is not an eigenvalue of T (and in particular if T has no eigenvalues), then $\lambda \notin W(T)$. It is observed that the eigenvalues of every operator T belong to $W(T)$. If $Tf = \lambda f$ with $\|f\| = 1$, then $\langle Tf, f\rangle = \lambda$. Since T is convexoid then $\|T\| = \sup\{|\lambda| : \lambda \in W(T)\}$ so that there always exists a $\lambda \in \overline{W(T)}$ such that $|\lambda| = \|T\|$. ■
Lemma 4.4. Let $T \in CX(H)$ then $W(T)$ is closed. Moreover, the algebraic numerical range of $s \in \mathcal{A}$ is contained in $\overline{W(T)}$ for any Banach algebra $\mathcal{A}$.
Proof: T is closed if $x \in H$ is a limit point of $D(T)$ {Domain of T} such that $D(T) \ni x_n \to x$ and $Tx_n \to y \in H$ then $x \in D(T)$ and $Tx = y$. Then $D(T) \ni x_n \to x \in$ H and $Tx_n \to y \in H$ then implies that $Tx = y$ where T is everywhere defined and bounded if $D(T) \ni x_n \to x$ implies $Tx_n \to y \in H$ and $Tx = y$. Let f_n be any sequence of vectors such that $f_n \in D(T)$ as $n \to \infty, f_n \to f$ and $Tf_n \to g$ then it follows that $f \in D(T)$ as $T(f) = g$. Since by proposition 4.1 T is bounded as shown above then it is closed.
Theorem 4.5. Let $\in CX(H)$. Then $\overline{W(T)} = \cap_{\alpha}\{\beta : |\beta - \alpha| = r(T - \alpha)\}$. Moreover, $r(T) = \|T\|$ if and only if T is self-adjoint.
Proof: Let $T \in CX(H)$ be a self-adjoint operator. Then $r(T) = \|T\|$ which gives

$r(T) \leq \{\|T\varphi\|\|\varphi\|: \varphi \in H, \|\varphi\| \leq 1\} \leq \|T\|$.
Recalling that
$\|T\| = \{|\langle T\varphi, \omega\rangle|: \varphi, \omega \in H, \|\varphi\|, \|\omega\| \leq 1\}$.
The goal is to change ω into φ in the above expression. We fix φ and $\omega \in H$ with $\|\varphi\| \leq 1$ and $\|\omega\| \leq 1$. We choose $\theta \in [0,2\pi)$ such that $|\langle T\varphi, \omega\rangle| = e^{i\theta}\langle T\varphi, \omega\rangle$. Let $\acute{\omega} := e^{-i\theta}\omega \in H$ so that $\|\acute{\omega}\| \leq 1$ and $\langle T\varphi, \acute{\omega}\rangle = |\langle T\varphi, \omega\rangle| \in \mathbb{R}$. By polarization identity we have;

$$\langle T\varphi, \omega\rangle = \frac{1}{4}(\langle T(\varphi + \acute{\omega}), \varphi + \acute{\omega}\rangle - \langle T(\acute{\varphi} - \acute{\omega}), \varphi - \omega\rangle + \imath\langle \acute{T}(\varphi + \imath\acute{\omega})\varphi + \imath\acute{\omega}\rangle - \imath\omega\langle T(\varphi - \imath\acute{\omega}),\ \varphi - \imath\acute{\omega}\rangle).$$

Following that T is self adjoint then the inner product is a real number and by our choice of $\acute{\omega}, \langle T\varphi, \acute{\omega}\rangle$ is real then the sum of the complex terms is zero [12]. Thus
$\langle T\varphi, \acute{\omega}\rangle = \frac{1}{4}(\langle T(\varphi + \acute{\omega}), \varphi + \acute{\omega}\rangle -$
$\langle T(\varphi, \acute{\omega}), \varphi - \acute{\omega}, \varphi - \acute{\omega}\rangle)$.

By definition of the numerical radius we have $|\langle T\rho, \rho\rangle| \leq \|\rho\|^2 r(T)$ for all $\rho \in H$. Whence
$|\langle T\varphi, \omega\rangle| \leq \frac{1}{4}(|\langle T(\varphi + \omega), \acute{\varphi} + \acute{\omega}\rangle| +$
$|\langle T(\varphi - \acute{\omega}), \varphi - \acute{\omega}\rangle|) \leq \frac{1}{4}r(T)(\|\varphi + \omega\|^2 +$
$\|\varphi + \omega\|^2)$
By parallelogram law we get that $|\langle T\varphi, \omega\rangle| \leq \frac{1}{4}r(T)(2\|\varphi\|^2 + 2\|\acute{\omega}\|^2) \leq r(T)$ as claimed. As $\varphi, \omega \in H$ were arbitrary elements with norm at most one, the expression for $\|T\|$ shows that $\|T\| \leq r(T)$.

Theorem 4.6. Let $T \in CX(H)$ be a triangle matrix such that a_{11} is 2 otherwise zero[13]. Then $W(T)$ is a closed disc. Moreover, $r(T)$ *is* 2.

Proof: Let $T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ then numerical range $(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1 \}$. If $T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ we find $Tx = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 0 \end{bmatrix}$. Then $\langle Tx, x\rangle = [2x_1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [2x_1\bar{x}_1] = 2x_1\bar{x}_1$. Given that $\|x\| = 1$ i.e $|x_1|^2 + |x_2|^2 = 1$ but $x_1\bar{x}_1 = |x_1|^2 = 1$. Hence $2x_1\bar{x}_1 = 2$. And $|\langle Tx, x\rangle| = |2x_1\bar{x}_1| = 2$.

Conclusions

These results are properties of numerical range of convexoid operators. The numerical range is a closed disc as seen in this work. It would be interesting to determine the numerical ranges of other operators that have not been studied. Furthermore, the relationship between numerical range and norm of convexoid operators could be looked at [14]. Certain properties of convexoid operators have been characterized for example continuity and linearity but numerical ranges and spectra of posinormal operators have not been considered. Also the relationship between the numerical range and spectrum has not been determined for convexoid operators. Therefore the objectives of this study were: to investigate numerical ranges of convexoid operator, to investigate the spectra of convexoid operators and to establish the relationship between the numerical range and spectrum of a convexoid operator. The methodology involved use of known inequalities like Cauchy- Schwartz inequality, Minkowski's inequality, the parallelogram law and the polarization identity to determine the numerical range and spectrum of convexoid operators and our technical approach shall involve use of tensor products. The results obtained shall be used in classification of Hilbert space operators and shall be applied in other fields like quantum information theory to optimize minimal output entropy of quantum channel; to detect entanglement using positive maps [15]; and for local distinguishability of unitary operators. The numerical range of a bounded convexoid operator A acting on a complex Hilbert space H is an ellipse whose foci are the eigenvalues of A. The results obtained are also useful in game theory and modeling [16].

Conflict of interest

Authors declare there are no conflicts of interest.

References

[1] Bauer FL. On the field of values subordinate to a norm. Numer Math. 1962;4:103-113.
[2] Furuta T. Some characteristics of convexoid operators. Rev Roum Maths Press et Appl. 1973;18:893-900.
[3] Furuta T, Nakamoto R. On the numerical range of an operator. Proc Japan Acad Soc. 1971;47:279-284.
[4] Halmos PR. A Hilbert space problem book. Von Nostrand Princeton, 1967.
[5] Lumer G. Semi inner product space. Trans Am Math Soc. 1961;100:29-43.
[6] Mecheri S. The numerical range of linear operators. Filomat. 2008;2:1-8.
[7] Okelo NB, Kisengo SK, Bonyo JO. Properties of Hilbert space operators.

Operator Theory. Lambert Academic Publishing, 2013.
[8] Seddik A. The numerical range of elementary operators II. Linear Algebra Appl. 2001;338:239-244.
[9] Seddik A. On numerical range and norm of elementary operators. linear Multilinear Algebra. 2004;52:293-302.
[10] Shapiro JH. Notes on the numerical range. Michigan State University, East Lansing, 2004.
[11] Toeplitz O. Das algebraische Analogon zu einem Satz von Feje'r. Math. Z. 1969;2:187-197.
[12] Vijayabalaji S, Shyamsundar G. Interval-valued intuitionistic fuzzy transition matrices. International Journal of Modern Science and Technology. 2016;1(2):47-51.
[13] Judith J O, Okelo NB, Roy K, Onyango T. Numerical Solutions of Mathematical Model on Effects of Biological Control on Cereal Aphid Population Dynamics. International Journal of Modern Science and Technology. 2016;1(4)138-143.
[14] Judith J O, Okelo NB, Roy K, Onyango T. Construction and Qualitative Analysis of Mathematical Model for Biological Control on Cereal Aphid Population Dynamics. International Journal of Modern Science and Technology. 2016;1(5)150-158.
[15] Vijayabalaji S, Sathiyaseelan N. Interval-Valued Product Fuzzy Soft Matrices and its Application in Decision Making, International Journal of Modern Science and Technology. 2016;1(6)159-163.
[16] Chinnadurai V, Bharathivelan K. Cubic Ideals in Near Subtraction Semigroups, International Journal of Modern Science and Technology. 2016;1(8)276-282.

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